DOI:10.1478/C1A0501006

Atti dell'Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
Vol. LXXXIII, C1A0501006 (2005)
Adunanza del 16 maggio 2005

MEASURE FOR FAMILIES OF HYPERPLANES SYSTEMS IN THE AFFINE SPACE

G. MOLICA BISCI*

(Nota presentata dal Socio Corrispondente Marius Stoka)

ABSTRACT. We study the problem of the measurability of families of k-hyperplanes having a common fixed point and placed in $\mathbb{A}^{2n+1}$, with $n \geq 3$. We prove that for $k = 2n$ or $k = 3$ and $2n = 3t$, for some $t \in \mathbb{N}$, the family is not measurable.

1. Introduction

Let $\mathbf{X}_n$ be an n-dimensional space, for example the affine or the projective space over the real field and

$$\mathbf{G}_r : \quad x_i' = f_i(x_1, ..., x_n; a_1, ..., a_r), \qquad (i = 1, ..., n), \tag{1}$$

a Lie group of transformations on $\mathbf{X}_n$ with $a_1, ..., a_r$ independent set of essential parameters for $\mathbf{G}_r$. We assume that the identity of the group $\mathbf{G}_r$ is determined by $a_1 = ... = a_r = 0$. A function ϕ, solution of the system of partial differential equations

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\xi_j^i(x_1, ..., x_n)\phi(x_1, ..., x_n)) = 0, \tag{2}$$

where

$$\xi_j^i(x_1, ..., x_n) = \frac{\partial x_i'}{\partial a_j}\Big|_{\mathbf{a}=\mathbf{0}}, \quad (j = 1, ..., r),$$

is an invariant integral function for the group $\mathbf{G}_r$. M.Stoka [8] defines measurable a group $\mathbf{G}_r$ with a unique invariant integral function ϕ, up to a constant factor. Let $\mathcal{V}_q$ be a family of p-dimensional varieties defined by

$$\mathcal{V}_q := \begin{cases} F_1(x_1, ..., x_n; \alpha_1, ..., \alpha_q) = 0 \\ \quad\quad ... \\ \quad\quad ... \\ F_{n-p}(x_1, ..., x_n; \alpha_1, ..., \alpha_q) = 0 \end{cases}$$

where $\alpha_s \in \mathbb{R}$, $s = 1, ..., q$ are essential independent parameters and F_k, $k = 1, ..., (n-p)$ are analytic functions on its domains. Following the definitions and the proof in [8], if $\mathbf{G}_r$ is the maximal invariant group of $\mathcal{V}_q$ one has an associated group of transformations

$$\mathbf{H}_r : \quad \alpha_h' = \varphi_h(\alpha_1, ..., \alpha_q; a_1, ..., a_r), \qquad (h = 1, ..., q),$$

and $\mathbf{G}_r$ and $\mathbf{H}_r$ are isomorphic.

If we suppose that $\mathbf{H}_r$ is measurable of invariant integral function $\phi(\alpha_1, ..., \alpha_q)$, the expressions

$$\mu_{\mathbf{G}_r}(\mathcal{V}_q) = \int_{\mathcal{F}(\alpha)} \phi(\alpha_1, ..., \alpha_q) d\alpha_1 \wedge ... \wedge d\alpha_q,$$

where $\mathcal{F}(\alpha) = \{(\alpha_1, ..., \alpha_q) \in \mathbb{R}^q | \ F_k(\mathbf{x}; \alpha_1, ..., \alpha_q) = 0, \ k = 1, ..., n-p\}$ and

$$|\phi(\alpha_1, ..., \alpha_q)| d\alpha_1 \wedge ... \wedge d\alpha_q,$$

are respectively the measure of the family $\mathcal{V}_q$ and the invariant density respect to the group $\mathbf{G}_r$ of the family $\mathcal{V}_q$. A family of varieties is measurable if there exist a unique measure ϕ. If the group $\mathbf{G}_r$ is not measurable this not implies that the family of varieties $\mathcal{V}_q$ is not measurable. We consider in this case, subgroups of the maximal invariant group $\mathbf{G}_r$. A subgroup of $\mathbf{H}_r$ is said of $\mathbf{H}_0$-type if it is isomorphic to a subgroup of $\mathbf{G}_r$ obtained with the conditions

$$\alpha_{l_1} = \alpha_{l_2} = ... = \alpha_{l_t} = 0,$$

for some t. A subgroup of $\mathbf{H}_r$ is $\mathbf{H}_0$-measurable if it is of $\mathbf{H}_0$-type and measurable. Its measure is said $\mathbf{H}_0$-measure. For other notions see [1]. Recently several authors studied problems of Integral Geometry in projective spaces, see for instance [9], [3], [4]. In this paper we take, as ambient space, an affine space $\mathbb{A}^{2n+1}$ over $\mathbb{R}$ of dimension $2n+1$, $n \geq 3$ and coordinates $x_1, ..., x_{2n+1}$. We consider a family of systems of k-hyperplanes, $1 \leq k < 2n+1$, passing for a fixed point that we assume the origin of coordinates. In terms of equations we have

$$x_1 + \sum_{j=2}^{2n+1} \beta_i^j x_j = 0, \quad 1 \leq i \leq k, \tag{3}$$

where $\beta_i^j \in \mathbb{R}$.

In [1] the authors showed that the family of systems of k-hyperplanes, $1 \leq k \leq 7$, having a common fixed point and placed in a 7-dimensional affine space, is not measurable for $k \neq 1, 5$ and it has a single $\mathbf{H}_0$-measure for $k = 1, 5$.

In this paper we attack the same problem for an affine space of higher dimension. We show that for $k = 2n$ and $k = 3$ with $2n = 3t$, for some $t \in \mathbb{N}$, the family of $2n$-hyperplanes, respectively 3-hyperplanes is not measurable. For the family of 1-hyperplanes we find the unique $\mathbf{H}_0$-measure

$$\phi = \frac{1}{\beta_1^2 \beta_1^3 \cdot ... \cdot \beta_1^{2n+1}}.$$

For other questions about this argument see the paper [2] or [5].

2. Preliminaries

The parameters space of the family of systems of k-hyperplanes in $\mathbb{A}^{2n+1}$ is of dimension $2nk$ and coordinates $\beta_i^j \in \mathbb{R}$ with $1 \leq i \leq k$ and $j = 2, ..., 2n+1$. The maximal group of invariance of the family is

$$\mathbf{G}_{(2n+1)^2}: \quad x_r = \sum_{s=1}^{2n+1} \alpha_r^s x_s^{'}, \quad r = 1, 2, ..., 2n+1 \tag{4}$$

with $\alpha_r^s \in \mathbb{R}$.

Acting by $\mathbf{G}_{(2n+1)^2}$ on (3) we obtain

$$x_1^{'} + \sum_{j=2}^{2n+1} \beta_i^{'j} x_j = 0, \quad 1 \leq i \leq k \tag{5}$$

$\beta_i^{'j} \in \mathbb{R}$, with

$$\mathbf{H}_{(2n+1)^2}: \qquad \beta_i^{'j} = \frac{\alpha_1^j + \sum_{l=2}^{2n+1} \beta_i^l \alpha_l^j}{\alpha_1^1 + \sum_{l=2}^{2n+1} \beta_i^l \alpha_l^1}, \tag{6}$$

where $1 \leq i \leq k$; $j = 2, ..., 2n+1$ and $\alpha_1^1 + \sum_{l=2}^{2n+1} \beta_i^l \alpha_l^1 \neq 0$.

The infinitesimal operators[1] are:

$$\xi_{11}^{ij} = -\beta_i^j; \quad \xi_{h1}^{ij} = -\beta_i^h \beta_i^j; \quad \xi_{1s}^{ij} = \delta_{js}; \quad \xi_{hs}^{ij} = \beta_i^h \delta_{js}, \tag{7}$$

with $1 \leq i \leq k$; $s, j, h = 2, 3, ..., 2n+1$ and where δ_{js} is the Kronecker's symbol.

Theorem 1. *The group $\mathbf{H}_{(2n+1)^2}$ given by*

$$\beta_i^{'j} = \frac{\alpha_1^j + \sum_{l=2}^{2n+1} \beta_i^l \alpha_l^j}{\alpha_1^1 + \sum_{l=2}^{2n+1} \beta_i^l \alpha_l^1}, \tag{8}$$

where $1 \leq i \leq k$, $j = 2, ..., 2n+1$ and $\alpha_1^1 + \sum_{l=2}^{2n+1} \beta_i^l \alpha_l^1 \neq 0$ is not measurable.

Proof: We consider the following Deltheil's system

$$\sum_{i=1}^{k} \sum_{j=2}^{2n+1} \beta_i^j \frac{\partial \phi}{\partial \beta_i^j} = -2nk\phi, \quad \sum_{i=1}^{k} \frac{\partial \phi}{\partial \beta_i^j} = 0, \quad j = 2, ..., 2n+1 \tag{9}$$

$$\sum_{i=1}^{k} \beta_i^j \sum_{h=2}^{2n+1} \beta_i^h \frac{\partial \phi}{\partial \beta_i^h} = -(2n-2)\phi \sum_{i=1}^{k} \beta_i^j, \quad j = 2, ..., 2n+1$$

---

[1]For the notions of Integral Geometry see [8]

$$\sum_{i=1}^{k} \beta_i^j \frac{\partial \phi}{\partial \beta_i^j} = -k\phi, \quad j = 2, ..., 2n+1$$

$$\sum_{i=1}^{k} \beta_i^j \frac{\partial \phi}{\partial \beta_i^h} = 0, \quad j, h = 2, ..., 2n+1; j \neq h$$

It is an easy remark that this system admits the unique trivial solution.

It is possible that the family of systems of k-hyperplanes is measurable. In fact for $k = 1$, we have the following

Theorem 2. *The family of systems of 1-hyperplanes having a common fixed point in* $\mathbb{A}^{2n+1}$ *is measurable with* $\mathbf{H}_0$*-measure given by*

$$\phi = \frac{1}{\beta_1^2 \beta_1^3 \cdot ... \cdot \beta_1^{2n+1}},$$

up to a constant non-zero factor.

Proof: We consider the subgroup $\mathbf{G}^{(1)}$ of $\mathbf{G}_{(2n+1)^2}$ given by

$$x_r = \alpha_r^r x_r^{'}, \quad r = 2, 3, ..., 2n+1$$

Denoting by $\mathbf{H}^{(1)}$ the correspondent subgroup in the parameters space $\mathbf{H}_{(2n+1)^2}$, we obtain the Deltheil's system

$$\sum_{j=2}^{2n+1} \beta_i^j \frac{\partial \phi}{\partial \beta_i^j} = -2n\phi, \quad \beta_1^j \frac{\partial \phi}{\partial \beta_1^j} = -\phi, \quad j = 2, 3, ..., 2n+1 \tag{10}$$

The unique (non-trivial) solution of this system is

$$\phi = \frac{1}{\beta_1^2 \beta_1^3 \cdot ... \cdot \beta_1^{2n+1}}.$$

Finally the subgroup $\mathbf{H}^{(1)}$ is $\mathbf{H}_0$-measurable.

3. Some non measurable family of hyperplanes systems

The main result of this section concern the non-measurability of two families of hyperplanes of the affine space. They can be described by the following Theorem 3 and Theorem 4.

Theorem 3. *The family of systems of* $2n$*-hyperplanes having a common fixed point in* $\mathbb{A}^{2n+1}$ *is not measurable.*

Proof: Taking the subgroup $\mathbf{G}^{(2)}$ of $\mathbf{G}_{(2n+1)^2}$, given by

$$x_1 = \alpha_1^1 x_1^{'} \quad x_r = \sum_{s=2}^{2n+1} \alpha_r^s x_s^{'}, \quad r = 2, 3, ..., 2n+1.$$

we denote by $\mathbf{H}^{(2)}$ the correspondent subgroup in the parameters space $\mathbf{H}_{(2n+1)^2}$. We can consider the following Deltheil's system

$$\sum_{i=1}^{2n}\sum_{j=2}^{2n+1} \beta_i^j \frac{\partial\phi}{\partial\beta_i^j} = -4n^2\phi, \tag{11}$$

$$\sum_{i=1}^{2n} \beta_i^j \frac{\partial\phi}{\partial\beta_i^j} = -2n\phi, \quad j = 2, ..., 2n+1$$

$$\sum_{i=1}^{2n} \beta_i^j \frac{\partial\phi}{\partial\beta_i^h} = 0, \quad j, h = 2, ..., 2n+1; j \neq h$$

The solution of the previous system is (up to a constant factor)

$$\phi = \frac{1}{[\det(M)]^{2n}},$$

where

$$M := \begin{pmatrix} \beta_{12} & \beta_{22} & \beta_{32} & \cdots & \cdots & \beta_{2n,2} \\ \beta_{13} & \beta_{23} & \beta_{33} & \cdots & \cdots & \beta_{2n,3} \\ \beta_{14} & \beta_{24} & \beta_{34} & \cdots & \cdots & \beta_{2n,4} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_{1,2n+1} & \beta_{2,2n+1} & \beta_{3,2n+1} & \cdots & \cdots & \beta_{2n,2n+1} \end{pmatrix}$$

Now we consider the subgroup $\mathbf{G}^{(3)}$ of $\mathbf{G}_{(2n+1)^2}$, given by

$$x_r = \sum_{s=2}^{2n+1} \alpha_r^s x_s^{'}, \quad r = 1, 2, 3, ..., 2n-1,$$

$$x_r = \alpha_r^r x_r^{'}, \quad r = 2n, 2n+1.$$

Denoting by $\mathbf{H}^{(3)}$ the correspondent subgroup in the parameters space $\mathbf{H}_{(2n+1)^2}$, as customary we give the Deltheil's system

$$\sum_{i=1}^{2n}\sum_{j=2}^{2n+1} \beta_i^j \frac{\partial\phi}{\partial\beta_i^j} = -4n^2\phi, \quad \sum_{i=1}^{2n} \frac{\partial\phi}{\partial\beta_i^j} = 0, \quad j = 2, ..., 2n+1 \tag{12}$$

$$\sum_{i=1}^{2n} \beta_i^j \sum_{h=2}^{2n+1} \beta_i^h \frac{\partial\phi}{\partial\beta_i^h} = -(2n+1)\phi \sum_{i=1}^{2n} \beta_i^j, \quad j = 2, ..., 2n+1$$

$$\sum_{i=1}^{2n} \beta_i^j \frac{\partial \phi}{\partial \beta_i^j} = -2n\phi, \quad j = 2, ..., 2n+1$$

$$\sum_{i=1}^{2n} \beta_i^j \frac{\partial \phi}{\partial \beta_i^h} = 0, \quad j = 2, 3, ..., 2n-1; h = 2, 3, ..., 2n+1; j \neq h$$

We obtain

$$\phi = \frac{\det[M_1] \cdot ... \cdot \det[M_{2n}]}{\det^{2n}[G_1] \det^{2n}[G_2]},$$

where

$$G_1 := \begin{pmatrix} 1 & \beta_1^2 & \beta_1^3 & \beta_1^4 & ... & \beta_1^{2n} \\ 1 & \beta_2^2 & \beta_2^3 & \beta_2^4 & ... & \beta_2^{2n} \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 1 & \beta_{2n-1}^2 & \beta_{2n-1}^3 & \beta_{2n-1}^4 & ... & \beta_{2n-1}^{2n} \\ 1 & \beta_{2n}^2 & \beta_{2n}^3 & \beta_{2n}^4 & ... & \beta_{2n}^{2n} \end{pmatrix}$$

$$G_2 := \begin{pmatrix} 1 & \beta_1^2 & \beta_1^3 & \beta_1^4 & ... & \beta_1^{2n-1} & \beta_1^{2n+1} \\ 1 & \beta_2^2 & \beta_2^3 & \beta_2^4 & ... & \beta_2^{2n-1} & \beta_2^{2n+1} \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... \\ 1 & \beta_{2n-1}^2 & \beta_{2n-1}^3 & \beta_{2n-1}^4 & ... & \beta_{2n-1}^{2n-1} & \beta_{2n-1}^{2n+1} \\ 1 & \beta_{2n}^2 & \beta_{2n}^3 & \beta_{2n}^4 & ... & \beta_{2n-1}^{2n-1} & \beta_{2n}^{2n+1} \end{pmatrix}$$

$$M_1 := \begin{pmatrix} 1 & \beta_2^2 & \beta_2^3 & \beta_2^4 & ... & \beta_2^{2n-1} \\ 1 & \beta_3^2 & \beta_3^3 & \beta_3^4 & ... & \beta_3^{2n-1} \\ ... & ... & ... & ... & ... & ... \\ 1 & \beta_{2n-1}^2 & \beta_{2n-1}^3 & \beta_{2n-1}^4 & ... & \beta_{2n-1}^{2n-1} \\ 1 & \beta_{2n}^2 & \beta_{2n}^3 & \beta_{2n}^4 & ... & \beta_{2n}^{2n-1} \end{pmatrix}$$

$$M_2 := \begin{pmatrix} 1 & \beta_1^2 & \beta_1^3 & \beta_1^4 & ... & \beta_1^{2n-1} \\ 1 & \beta_3^2 & \beta_3^3 & \beta_3^4 & ... & \beta_3^{2n-1} \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 1 & \beta_{2n-1}^2 & \beta_{2n-1}^3 & \beta_{2n-1}^4 & ... & \beta_{2n-1}^{2n-1} \\ 1 & \beta_{2n}^2 & \beta_{2n}^3 & \beta_{2n}^4 & ... & \beta_{2n}^{2n-1} \end{pmatrix}$$

.
.
.

and

$$M_{2n} := \begin{pmatrix} 1 & \beta_1^2 & \beta_1^3 & \beta_1^4 & ... & \beta_1^{2n-1} \\ 1 & \beta_2^2 & \beta_2^3 & \beta_2^4 & ... & \beta_2^{2n-1} \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 1 & \beta_{2n-2}^2 & \beta_{2n-2}^3 & \beta_{2n-2}^4 & ... & \beta_{2n-2}^{2n-1} \\ 1 & \beta_{2n-1}^2 & \beta_{2n-1}^3 & \beta_{2n-1}^4 & ... & \beta_{2n-1}^{2n-1} \end{pmatrix}$$

Finally we consider now the case of family of systems of 3-hyperplanes.

Theorem 4. *The family of systems of* 3*-hyperplanes having a common fixed point in* $\mathbb{A}^{2n+1}$ *with* $2n = 3t$*, for some* $t \in \mathbb{N}$*, is not measurable.*

Proof: Let us consider the subgroup $\mathbf{G}^{(4)} \subset \mathbf{G}_{(2n+1)^2}$ given by

$$x_r = \alpha_r^r x_r^{'}, \quad r = 2, 3..., , 2n$$

$$x_r = \sum_{s=1}^{2n+1} \alpha_r^s x_s^{'}, \quad r = 1, 2n+1.$$

Let us denote by $\mathbf{H}^{(4)}$ the correspondent subgroup in the parameters space $\mathbf{H}_{(2n+1)^2}$. The Deltheil's system will be

$$\sum_{i=1}^{3}\sum_{j=2}^{2n+1} \beta_i^j \frac{\partial \phi}{\partial \beta_i^j} = -6n\phi, \quad \sum_{i=1}^{3} \frac{\partial \phi}{\partial \beta_i^j} = 0, \quad j = 2, ..., 2n+1 \tag{13}$$

$$\sum_{i=1}^{3} \beta_i^{2n+1} \sum_{h=2}^{2n+1} \beta_i^h \frac{\partial \phi}{\partial \beta_i^h} = -(2n+1)\phi \sum_{i=1}^{3} \beta_i^{2n+1},$$

$$\sum_{i=1}^{3} \beta_i^j \frac{\partial \phi}{\partial \beta_i^j} = -3\phi, \quad j = 2, ..., 2n+1$$

$$\sum_{i=1}^{3} \beta_i^{2n+1} \frac{\partial \phi}{\partial \beta_i^h} = 0, \quad h = 2, 3, ..., 2n.$$

Hence, after some calculations the solution is

$$\phi = \frac{\det^4[D_{2n}] \det^4[D_{2n+1}] \det^4[D_{2n+2}]}{\det^3[D_1] \det^3[D_2] \cdot ... \cdot \det^3[D_{2n-1}]},$$

where

$$D_1 := \begin{pmatrix} 1 & 1 & 1 \\ \beta_1^2 & \beta_2^2 & \beta_3^2 \\ \beta_1^{2n+1} & \beta_2^{2n+1} & \beta_3^{2n+1} \end{pmatrix} \quad D_2 := \begin{pmatrix} 1 & 1 & 1 \\ \beta_1^3 & \beta_2^3 & \beta_3^3 \\ \beta_1^{2n+1} & \beta_2^{2n+1} & \beta_3^{2n+1} \end{pmatrix}$$

$$D_3 := \begin{pmatrix} 1 & 1 & 1 \\ \beta_1^4 & \beta_2^4 & \beta_3^4 \\ \beta_1^{2n+1} & \beta_2^{2n+1} & \beta_3^{2n+1} \end{pmatrix} \quad D_4 := \begin{pmatrix} 1 & 1 & 1 \\ \beta_1^5 & \beta_2^5 & \beta_3^5 \\ \beta_1^{2n+1} & \beta_2^{2n+1} & \beta_3^{2n+1} \end{pmatrix}$$

$$\vdots$$

$$D_{2n-1} := \begin{pmatrix} 1 & 1 & 1 \\ \beta_1^{2n} & \beta_2^{2n} & \beta_3^{2n} \\ \beta_1^{2n+1} & \beta_2^{2n+1} & \beta_3^{2n+1} \end{pmatrix}$$

$$D_{2n} := \begin{pmatrix} 1 & 1 \\ \beta_1^{2n+1} & \beta_2^{2n+1} \end{pmatrix}$$

$$D_{2n+1} := \begin{pmatrix} 1 & 1 \\ \beta_1^{2n+1} & \beta_2^{2n+1} \end{pmatrix} \quad D_{2n+2} := \begin{pmatrix} 1 & 1 \\ \beta_1^{2n+1} & \beta_3^{2n+1} \end{pmatrix}$$

If $\mathbf{G}^{(5)}$ is the subgroup of $\mathbf{G}_{(2n+1)^2}$, given by

$$x_1 = \alpha_1^1 x_r x^{'}, \quad x_r = \sum_{s=2}^{2n-2} \alpha_r^s x_s^{'}, \quad r = 2, 3, ..., 2n-2,$$

$$x_r = \sum_{s=1}^{2n-1} \alpha_r^s x_s^{'}, \quad r = 2n-1, 2n, 2n+1.$$

and $\mathbf{H}^{(5)}$ is the correspondent subgroup in the parameters space $\mathbf{H}_{(2n+1)^2}$ we have the system

$$\sum_{i=1}^{3} \sum_{j=2}^{2n+1} \beta_i^j \frac{\partial \phi}{\partial \beta_i^j} = -6n\phi \tag{14}$$

$$\sum_{i=1}^{3} \beta_i^j \frac{\partial \phi}{\partial \beta_i^h} = -3\phi, \quad j = 2, ..., 2n+1$$

$$\sum_{i=1}^{3} \beta_i^j \frac{\partial \phi}{\partial \beta_i^h} = 0,$$

with $j, h = 2, ..., 2n-2, \; h \neq j; j, h = 2n-1, 2n, 2n+1, \; h \neq j.$

Hence the solution is

$$\phi = \frac{1}{\det^3[B_1]\det^3[B_2] \cdot ... \cdot \det^3[B_{\frac{2n}{3}}]},$$

where

$$B_1 := \begin{pmatrix} \beta_1^2 & \beta_1^3 & \beta_1^4 \\ \beta_2^2 & \beta_2^2 & \beta_2^4 \\ \beta_3^2 & \beta_3^3 & \beta_3^4 \end{pmatrix} \quad B_2 := \begin{pmatrix} \beta_1^5 & \beta_1^6 & \beta_1^7 \\ \beta_2^5 & \beta_2^6 & \beta_2^7 \\ \beta_3^5 & \beta_3^6 & \beta_3^7, \end{pmatrix}$$

$$\vdots$$

$$B_{\frac{2n}{3}-1} := \begin{pmatrix} \beta_1^{2n-4} & \beta_1^{2n-3} & \beta_1^{2n-2} \\ \beta_2^{2n-4} & \beta_2^{2n-3} & \beta_2^{2n-2} \\ \beta_3^{2n-4} & \beta_3^{2n-3} & \beta_3^{2n-2} \end{pmatrix} \quad B_{\frac{2n}{3}} := \begin{pmatrix} \beta_1^{2n-1} & \beta_1^{2n} & \beta_1^{2n+1} \\ \beta_2^{2n-1} & \beta_2^{2n} & \beta_2^{2n+1} \\ \beta_3^{2n-1} & \beta_3^{2n} & \beta_3^{2n+1} \end{pmatrix}$$

Remark We observe that for $n = 3$ and $1 \leq k \leq 7$ the family of systems of k-hyperplanes having a common fixed point and placed in a seven dimensional affine space is not measurable for $k \neq 1, 5$ and has a single $\mathbf{H}_0$-measure for $k = 1$ or $k = 5$, as showed in [1].

References

[1] P. Dulio, M. Petriccione, "Sulla misurabilità delle famiglie dei sistemi di k iperpiani passanti per un punto fisso dello spazio affine $\mathbb{A}_7$", *Atti Accad. Pelorit. Pericol. - Cl. Sci. Fis. Mat. Nat.* **LXXI**, 407 (1995).

[2] P. Dulio, M. Petriccione, "Sulla misurabilità delle famiglie dei sistemi di due e tre iperpiani passanti per un punto fisso di $\mathbb{A}_4$", *Atti Accad. Pelorit. Pericol. - Cl. Sci. Fis. Mat. Nat.* **LXXI**, 427 (1995).

[3] A. Duma, M. Stoka, "On the measurability of the conic sections' family in the projective space $\mathbb{P}^3$", *Rend. Circ. Mat. Palermo*, Serie **II**, Tomo LI, 537-548 (2002).

[4] G. Failla, G. Molica Bisci,"On the measurability of the conic sections' family in the projective space $\mathbb{P}^4$", *Bull. U.M.I.*, to appear.

[5] G. Santoro, "Sulla misurabilità della famiglia di superfici cubiche spezzate in tre piani passanti per un punto fisso", *Atti Accad. Pelorit. Pericol. - Cl. Sci. Fis. Mat. Nat.* **LXVIII**, 269 (1992).

[6] G. Santoro, "Sulla misurabilità di una famiglia di superfici cubiche dello spazio affine A_3", *Atti Accad. Pelorit. Pericol. - Cl. Sci. Fis. Mat. Nat.* **LXVIII**, 257 (1992).

[7] M. Stoka, "Masura familiior de varietati din spatiul E_3", *St. Cerc. Mat.*, **IX**, 547-558 (1958).

[8] M. Stoka, "Géométrie Intégrale", *Mém. Sci. Math.*, **165** (Gauthier-Villars, Paris, 1968).

[9] M. Stoka, "On the measurability of the conic sections' family in the projective space $\mathbb{P}^n$", *Bull. U.M.I.* (**8**) 7-B (2004), 357-367.

Giovanni Molica Bisci
University of Reggio Calabria,
Faculty of Engineering, DIMET.
Via Graziella (Feo di Vito)
I-89100 Reggio Calabria, Italy
* **E-mail**: giovanni.molica@ing.unirc.it

Presented: May 16, 2005
Published on line on November 29, 2005