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ON THE k-UNIRATIONALITY OF THE CUBIC COMPLEX

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ABSTRACT. We show that the complete intersection $V = V(2,3) \subseteq \mathbb{P}^5$ of a quadric and a cubic in 5-dimensional projective space defined over a field k, of char. $\neq 2, 3$, is unirational over this field k itself if moreover V has a point p rational over k and if one of the two planes through p on the quadric is also rational over k.

1. Introduction

In 1912 Enriques [1] showed that the complete intersection $V(2,3) \subseteq \mathbb{P}^5$ of a quadric and a cubic in 5-dimensional projective space (the cubic complex) is unirational. In Enriques proof the base field is not explicitly mentioned, but it is implicitly assumed to be the complex field. In 1938 Morin [3] remarked, without proof, that the irrationalities introduced in Enriques proof depend only from the determination of one point and of one of the two families of 2-planes of the quadric through the $V(2,3)$. He needed this in order to prove that the generic quintic hypersurface $V(5) \subseteq \mathbf{P}^r$ in r-dimensional projective space is unirational as soon as $r \geq 17$. In this paper we give a proof in details of Morin's remark. In doing this we follow closely Enriques construction, our only contribution being to fully explain and justify his statements. In a forthcoming paper we will apply this result in order to clarify, in the same spirit, Morin's theorem on the quintic.

2. The Main Theorem

We are interested in the field over which the variety is unirational. Recall that a variety is k-unirational, or unirational over k, if it is unirational and if moreover the rational dominant map from the projective space to the variety is defined over the field k itself.

Theorem 2.1. (Enriques) *Let $V = V_3(3,2) = C \cap Q \subset \mathbb{P}^5$ be defined over a field k (char. $k \neq 2, 3$) with the quadric Q and V itself non-singular.*
Assume moreover that
1) $\exists\ p_0 \in V(k)$ *(i.e. a point rational over k);*
2) *one 2-plane Λ' on Q through p_0 is also defined over k.*
Then V is k-unirational.

Remark 2.2. a) If the assumptions 1) and 2) are not satisfied then of course a finite extension of k suffices.
b) Note that 1) and 2) are precisely the conditions used by Morin.

Proof.
First consider the quadric $Q = Q_4 = V_4(2) \subset \mathbb{P}^5$. Let the projective coordinates in $\mathbb{P}^5$ be $(x_0, x_1, x_2, y_0, y_1, y_2)$. We can assume that:
$p_0 = (1, 0, ..., 0)$,
Λ' is given by $y_0 = y_1 = y_2 = 0$.
Then by a projective transformation of coordinates *over k* we can arrange that the equation of Q is given by

$$x_0y_0 + x_1y_1 + x_2y_2 = 0 \tag{1}$$

(See for istance [2], page 226.)
Note that the tangent space to Q in p_0, the $T_{p_0}(Q)$, is given by $y_0 = 0$.

Lemma 2.3. *Let $Q_4 \subset \mathbb{P}^5$ be given by* (1)*. Let $p = (1, a_1, a_2, b_0, b_1, b_2)$ be an arbitrary point on Q.*
Then there are two 1-dimensional families $W'(p), W''(p)$ of 2-planes on Q through p and $W'(p)$ (resp. $W''(p)$) are rational over $K = k(p)$.

Proof.
Let $H_\infty := \{x_0 = 0\}$ and $Q_3^\infty = Q_4 \cap H_\infty$; this is a quadric which is in fact *a cone* with vertex $(0, 0, 0, 1, 0, 0)$ over the *quadric surface* $Q_2(p_0)$ with equation

$$x_1y_1 + x_2y_2 = 0 \tag{2}$$

in $H_\infty \cap T_{p_0}(Q) = \{x_0 = y_0 = 0\}$.
On $Q_2(p_0)$ we have two ∞^1-families of lines. Now consider the tangent space $T_p(Q_4)$ in p to Q_4. It has as equation (over $K = k(p)$)

$$b_0x_0 + b_1x_1 + b_2x_2 + y_0 + a_1y_1 + a_2y_2 = 0. \tag{3}$$

Consider the *quadric surface* $Q_2(p)$ in $H_\infty \cap T_p(Q_4) \cap Q_4 = Q_3^\infty \cap T_p(Q_4)$; it has equations ($x_0 = 0$) + (3) + (2) and carries two 1-dim. families $\mathbf{F}', \mathbf{F}''$ of lines given by the equations $x_0 = 0$ and (3) and:

$$(4) \qquad \begin{cases} y_2 = \lambda x_1 \\ y_1 = -\lambda x_2 \end{cases} \text{ resp. } \begin{cases} x_2 = \mu x_1 \\ y_1 = -\mu y_2 \end{cases}$$

the two families $W'(p), W''(p)$ of 2-planes on Q_4 through p are given as *spans*

$$\Lambda'_\lambda(p) = \langle p, l'_\lambda \rangle, l'_\lambda \in \mathbf{F}' \text{ resp. } \Lambda''_\mu(p) = \langle p, l''_\mu \rangle, l''_\mu \in \mathbf{F}''$$

and they have equations (3) + (4).
Therefore $W'(p) - \overset{\sim}{-} -> \mathbb{P}^1$, over $K = k(p)$, (resp. $W''(p) - \overset{\sim}{-} -> \mathbb{P}^1$, over $K = k(p)$), via the parameter λ (resp. μ); hence they are *rational curves*, *rational over their field of definition* $K = k(p)$.

Starting now with $p_0 \in V$ one constructs a rational curve, say $A(p_0)$ on V as follows: first take the family $W'(p_0)$ of 2-planes on Q_4 through p_0, next consider in each $\Lambda'_\lambda(p_0)$ of $W'(p_0)$ the cubic curve $B_\lambda(p_0) = \Lambda'_\lambda(p_0) \cap C$ and then take the "third" point $R_\lambda(p_0)$ of the intersection of $B_\lambda(p_0)$ with the tangent line to $B_\lambda(p_0)$ in p_0. In this way one gets, starting with p_0, first an "abstract" curve

$$(5) \qquad A^*(p_0) = \{R_\lambda(p_0); \lambda\} - \overset{\sim}{-} -> \mathbb{P}^1 \text{ over } k(p_0) = k$$

and a morphism $\varphi_0 : A^*(p_0) \longrightarrow V$ defined over $k(p_0) = k$, by $\varphi_0(\lambda) = R_\lambda(p_0)$, giving the "concrete" curve $\varphi_0(A^*(p_0)) =: A(p_0) \subset V$ (again rational over $k(p_0) = k$).

Now repeat the construction: take on $A(p_0)$ the generic point $p_1 = R_\lambda(p_0)$ (i.e. λ transcendental over k) and construct $A^*(p_1)$ and $A(p_1)$. Let

$$S^*(p_0) := S^* := \{A^*(p) | p \in A^*(p_0)\} - \overset{\sim}{-} -> \mathbb{P}^2 \text{ over } k \text{ (here we use Lemma } \mathbf{2.3})$$

and we have a morphism $\varphi_1 : S^* \longrightarrow V$; put $S = Im(\varphi_1) \subset V$. So S is the Zariski closure over k of the point $R_{\lambda_1}(p_1)$ in V with λ_1 transcendental over $k(\lambda)$ with $p_1 = R_{\lambda_1}(p_0)$, with λ transcendental over k.
Repeat the construction once more by taking $p_2 = R_{\lambda_1}(p_1)$ and let

$$V^*(p_0) := V^* = \{A^*(p) | p \in S^*(p_0)\} - \overset{\sim}{-} -> \mathbb{P}^3$$

over k (we use again Lemma **2.3**; see Remark **2.4** below) and $\varphi_2 : V^* \longrightarrow V$ and put $\widetilde{V} = Im(\varphi_2) \subset V$ all over k. Clearly $\widetilde{V}$ *is unirational over* k.

Claim. $\widetilde{V} = V$ *i.e.* φ_2 *is surjective.*

From this claim follows then immediately the theorem.

Remark 2.4. Of course we have used Lemma **2.3** because when we repeat the construction we get instead of (5)

$$A^*(p_1) = \{R_{\lambda_1}(p_0); \lambda_1\} - \overset{\sim}{-} -> \mathbb{P}^1 \tag{6}$$

over $k(\lambda) = k(p_1)$ by the lemma and this we need for the rationality (over k) of S^*; similar for V^*.

Proof of the claim.

Step 1 $S = Im\varphi_1$ is a surface.

Proof. S is the Zariski closure of $R_{\lambda_1}(p_1)$ *over* k, i.e. the Zariski closure *over* k of the curve $A(p_1) = ImA^*(p_1)$ (which is a *curve over* $k(p_1)$). Assume to the contrary that S is only a curve; then it is the curve $A(p_1)$ itself.
In order to see that this leads to a contradiction we need first:

Lemma 2.5. *$p_0 \in A(p_0)$ and is a multiple point (of multiplicity at least 4).*

Proof. We return to the notations in Lemma **2.3** and we "compute" the point $R_\lambda(p_0) \in \Lambda'_\lambda(p_0)$ of the curve $A(p_0)$.
In $\Lambda'_\lambda(p_0)$ we can use as non-homogeneous coordinates x_1 and x_2 and $y_2 = \lambda x_1, y_1 = -\lambda x_2$.
On the other hand let

$$f(x_1, x_2, y_0, y_1, y_2) = 0 \tag{7}$$

be the *non-homogeneous* equation of the cubic C, then the equation of the cubic curve $C \cap \Lambda'_\lambda(p_0) = B_\lambda(p_0)$ has the form

$$\begin{aligned} &f(x_1, x_2, 0, -\lambda x_2, \lambda x_1) = \\ &= l(x_1, x_2, -\lambda x_2, \lambda x_1) + q(x_1, x_2, -\lambda x_2, \lambda x_1) + c(x_1, x_2, -\lambda x_2, \lambda x_1) = 0 \end{aligned} \tag{8}$$

where l, q and c are linear, quadratic and cubic respectively with coefficients in k. (No constant term because $p_0 \in C \cap \Lambda'_\lambda(p_0)$). We can put:

$$l(x_1, x_2, -\lambda x_2, \lambda x_1) = (\alpha_1 + \lambda\beta_1)x_1 + (\alpha_2 + \lambda\beta_2)x_2$$

(with $\alpha_1, \alpha_2, \beta_1, \beta_2$ in k); $(\alpha_1 + \lambda\beta_1)x_1 + (\alpha_2 + \lambda\beta_2)x_2 = 0$ is the equation of the tangent line of $C \cap \Lambda'_\lambda(p_0)$ in p_0.
Hence we find the intersection points with the tangent line in p_0 by substituting

$$x_2 = -\frac{\alpha_1 + \beta_1\lambda}{\alpha_2 + \beta_2\lambda}x_1$$

in (8) which finally gives

$$0 = q(x_1, -\frac{\alpha_1 + \beta_1\lambda}{\alpha_2 + \beta_2\lambda}x_1, +\lambda\frac{\alpha_1 + \beta_1\lambda}{\alpha_2 + \beta_2\lambda}x_1, \lambda x_1) + c(...)$$

This gives

$$x_1^2 q^*(\lambda) + x_1^3 c^*(\lambda) = 0$$

where $q^*(\lambda)$ is non-homogeneous of degree 4 in λ and $c^*(\lambda)$ is non-homogeneous of degree 6 in λ, which has as solutions:
2 times $x_1 = 0$ (i.e. the tangency point p_0 counted twice) and

$$x_1 = -\frac{q^*(\lambda)}{c^*(\lambda)} \tag{9}$$

which finally gives the point $R_\lambda(p_0)$ we are looking for. Now we get $R_\lambda(p_0) = p_0$ itself if

$$q^*(\lambda) = 0, \tag{10}$$

i.e. for 4 values of λ. This means that $p_0 \in A(p_0)$ and it is infact a 4-multiple point; which we see if we intersect $A(p_0)$ with a general linear space

$$L(x_1, x_2, y_0, y_1, y_2) = 0$$

through p_0. By substituing (9) in $L = 0$ we get from the homogeneous L the equation for λ as follows

$$q^*(\lambda)L(...\lambda...) = 0.$$

So we get p_0 4-times from (10) and Lemma (**2.5**) is proved.

Returning to the proof of *Step 1* (i.e. S is a surface) we have that in case S is a curve it must be the curve $A(p_1)$ because the Zariski closure *over* $k(p_1)$ is already $A(p_1)$ (i.e. already a curve).
However now from Lemma **2.5** (applied to p_1, and over the field $K = k(p_1)$) it follows that $p_1 \in A(p_1)$. However then we must have

$$A(p_1) = A(p_0) \tag{11}$$

because the Zariski closure of p_1 over k is $A(p_0)$ and if $p_1 \in A(p_1)$, this Zariski closure over k is in the Zariski closure of $R_{\lambda_1}(p_1) = p_2$ over k.
Hence $S = A(p_1) = A(p_0)$, but this is impossible because p_1 is a multiple point on $A(p_1)$ by Lemma **2.5** and generic on $A(p_0)$. This is a *contradiction*, hence S *is a surface*.

Step 2 $\widetilde{V} \subseteq V$ *is a threefold, i.e.* $\widetilde{V} = V$.

Proof. Let $P \in V$ be a generic point. Now we want to find a point $R \in S$ such that $P \in A(R)$ (because $\widetilde{V} = \bigcup_{R \in S} A(R)$).
Let $\Gamma(P)$ = union of 2-planes on Q_4 through P. Clearly $\Gamma(P) \subset Q_4$ and $\Gamma(P)$ is a 3-dimensional subvariety on Q_4. By Lefschetz hyperplane theorem

$$\mathbb{Z} \simeq H^2(\mathbb{P}^r) \simeq H^2(Q_4)$$

hence $\Gamma(P)$ is, as cohomological class, a (positive) multiple of the hyperplane section hence $S \cap \Gamma(P) = D(P)$ is a *curve* on Q_4 (and hence a curve on $S \subset V$). On the other hand let $\mathbf{P}_P(C)$ be the polar variety (in $\mathbb{P}^5$) of the point P of the cubic C_4. Then $D(P) \cap \mathbf{P}_P(C) \neq \emptyset$, and for $R \in D(P) \cap \mathbf{P}_P(C)$ we have that $A(R)$ passes through P. This completes the proof.

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