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AN INTRODUCTION TO ENDOREVERSIBLE THERMODYNAMICS

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ABSTRACT. Reversible thermodynamic processes are convenient abstractions of real processes, which are always irreversible. Approaching the reversible regime means to become more and more quasistatic, letting behind processes which achieve any kind of finite transformation rate for the quantities studied. On the other hand studying processes with finite transformation rates means to deal with irreversibilities and in many cases these irreversibilities must be included in a realistic description of such processes. Endoreversible thermodynamics is a non-equilibrium approach in this direction by viewing a system as a network of internally reversible (endoreversible) subsystems exchanging energy in an irreversible fashion. This material provides an introduction to the subject.

1. Introduction

Equilibrium thermodynamics compares real processes to reversible processes proceeding without losses at an infinite slow speed. An example is the often used Carnot [1] efficiency:

$$\eta_{\mathrm{C}} = 1 - T_{\mathrm{L}}/T_{\mathrm{H}}\,. \tag{1}$$

It gives the fraction of the heat which at most can be converted to work in any engine using heat from a hot reservoir at temperature T_{H} and rejecting some of the heat to a reservoir at lower temperature T_{L}. Although the performance limits of reversible processes like the Carnot efficiency provide upper bounds for real irreversible processes they may not be good enough to be a useful guide in the improvement of real processes. Real heat engines, for example, seldom attain more than a fraction of the reversible Carnot efficiency.

Engineers tried to close this discrepancy between real process and limiting reversible process by improving their design, specific to certain devices or processes. But despite all technological progress in engineering, the gap remains, and it has to remain due to the irreversible nature of real processes.

Thus the principle question remains:

> *Under which conditions and how can realistic bounds for process variables of thermodynamic processes, which are performed in finite time, be determined*

and

> *What are the optimal process paths to achieve the optimal process values.*

This challenge has eventually inspired scientists to conduct a wide spectrum of research activities.

In the late 50's of the 20th century, the German H. Müser considered the power output of solar cells under the condition of a finite, irreversible, radiative energy transfer [2] and the Russians Novikov [3, 4] and Vukalovich [5] as well as the French Chambadal [6, 7] investigate the effect of finite heat transfer on the power output of an otherwise reversible power plant. They discovered that the efficiency at the maximum power point,

$$\eta = 1 - \sqrt{T_{\mathrm{L}}/T_{\mathrm{H}}}\ , \tag{2}$$

is considerably lower than the corresponding Carnot efficiency (1). In 1975 Curzon and Ahlborn [8] re-discovered the efficiency expression (2) and found it in remarkable agreement with the performance data of real power plants.

Since then a number of non-equilibrium thermodynamic theories were developed, for instance finite-time thermodynamics, which aims at capturing all effects due to processes occurring in finite-time or with finite rates [9]. Another example is endoreversible thermodynamics where the term 'endoreversible' means internally reversible and goes back to Rubin [10, 11]. The aim [12, 13] of all these investigations is to identify the main thermodynamic features of the system in order to make a model as simple as possible. The idea is to reduce the mathematical description and computational effort and yet to find more realistic optima and bounds for the operation of a thermodynamical system.

The concept of 'endoreversibility' has proven to be a powerful tool for the construction of models with the desired qualities. Endoreversible systems [10, 14] basically are composed of internally reversible subsystems with (irreversible) interactions between them. The losses due to the finite times or rates of processes are located in the interactions alone. The hypothesis of endoreversibility simplifies the expenditure for the analysis essentially. It has been successfully applied to a wide variety of thermodynamic systems and lead to remarkable results [15].

An important question in the analysis of endoreversible systems is how to deal with the time dependence of process variables and parameters, i.e. how the dynamics of a system evolves during a process.

The simplest systems are those where all flows and intensities during a process are constant though still finite. Such systems often allow an analytical solution for their performance characteristics and for their optimal points of operation.

More complicated systems permit a switching between several quasi-static regimes. An example for this category is the model of a heat engine which alternately connects to heat reservoirs at respective high and low temperatures [8]. The switching parameters, here the contact times, are usually externally controllable and can be optimized with respect to a certain objective, such as the power output of the heat engine.

Finally, systems can be investigated which require to consider the full dynamics of a process. In this case, the process evolves along a path which is undetermined, but may be subject to some constraints and bounds such as initial and final conditions. Determining performance limits for such systems leads to a path optimization problem. One has to find that thermodynamic path which extremalizes a given performance measure. Take, for example, the problem of optimizing the piston movement for an internal combustion Diesel engine such that the power output is maximized. The only controllable variable of the system is the volume of the cylinder which can be adjusted by moving the piston. Solving this

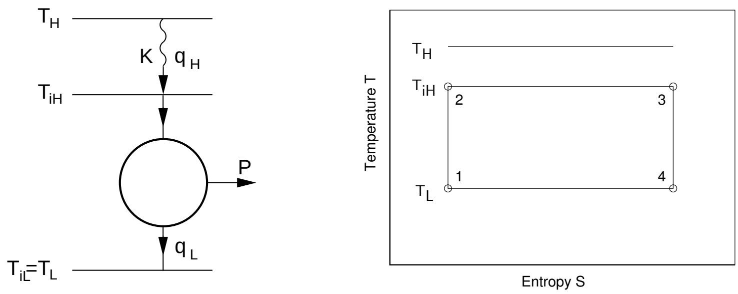

FIGURE 1. Model of the endoreversible Novicov engine with finite heat conduction K to the high temperature heat reservoir *(left)*. TS-diagram of a Carnot cycle with a temperature difference to the high temperature heat reservoir *(right)*.

optimization problem should give us the optimal piston movement, the resulting optimal work and possibly some other process variables such as temperature and pressure.

This review is a combination of material from two articles published previously [15, 16], and gives an overview on endoreversible techniques and systems. For a more complete description the reader should turn to these two articles.

2. Endoreversibility and endoreversible systems

2.1. Definition of the endoreversible subsystems. An endoreversible system consists of a number of subsystems which interact with each other and with their surroundings. We choose the *subsystems* so as to insure that each one undergoes only reversible processes. All the dissipation or irreversibility occurs in the *interactions* between the subsystems or the surroundings. An endoreversible system is thus defined by the properties of its subsystems and of its interactions. We call processes of such systems endoreversible process.

If a subsystem is for instance a spatially uniform working fluid, than the requirement that it undergoes only reversible processes means that it is always in internal thermodynamic equilibrium. But subsystems can be also more aggregate objects, namely engines (or more general energy transformation devices). If for instance such an engine takes in heat at temperature T_{H} and converts it into work and heat discharged at temperature T_{L}, then endoreversibility requires its efficiency η to be the Carnot efficiency $\eta_{\mathrm{C}} = 1 - T_{\mathrm{L}}/T_{\mathrm{H}}$. This will become more apparent in our first example.

2.2. An introductory example. As a simple introductory example we consider the Novicov engine [3, 4], a simplified version of the Curzon-Ahlborn engine treated later. The Novicov engine is a continuously operating, reversible Carnot engine with the internal temperatures T_{iH} and T_{iL}. It is in direct contact with the external low temperature heat bath at T_{L} and is coupled to an external high temperature heat bath at T_{H} through a finite heat conductance K (see Figure 1). The heat baths are both considered to be infinite such that the in- and outflux of energy does not change their temperatures.

The question is now how the irreversibility introduced by the finite heat conductance influences the performance of the engine. Does it for instance have an effect on the efficiency of the system?

We first note that the total heat flux through the system is limited, and thus the power produced by the engine is limited as well. To characterize the performance of this endoreversible system in more detail we want to determine the maximum power available and the efficiency at the operating point of maximum power.

Due to the finite heat conductance heat is only transported to the Carnot engine, if its high temperature T_{iH} is lower than the bath temperature T_{H}. The heat flux q_{H} transported from the heat bath to the engine is assumed to be proportional to the temperature difference (Newtonian heat conduction)

$$q_{\mathrm{H}} = K(T_{\mathrm{H}} - T_{\mathrm{iH}}). \tag{3}$$

At the low temperature side the heat can be discharged to the heat bath at T_{L} without any resistance. Thus this interaction between heat bath and engine is characterized not by a transfer law but by the requirement that the lower temperature T_{iL} of the Carnot engine is the same as the bath temperature

$$T_{\mathrm{L}} = T_{\mathrm{iL}}\ . \tag{4}$$

We note that the Carnot engine is characterized by three energy fluxes and two temperatures: q_{H} enters the engine at temperature T_{iH}, the heat flux to the low temperature heat bath q_{L} leaves the engine at temperature T_{iL}, and P is the power delivered by the engine. The heat baths are described by $(T_{\mathrm{H}}, q_{\mathrm{H}})$ and $(T_{\mathrm{L}}, q_{\mathrm{L}})$ respectively, and the heat conduction contains the parameter K. All the variables (energy fluxes, temperatures, and K) are related by the interaction between the heat bath and the engine and by the constraints coming from the endoreversibility of the Carnot engine. As the engine operates continuously in a steady state, all the energy fluxes have to balance

$$0 = q_{\mathrm{H}} - q_{\mathrm{L}} - P. \tag{5}$$

In addition, as the engine operates reversibly the entropy fluxes to and from the engine have to cancel

$$0 = \frac{q_{\mathrm{H}}}{T_{\mathrm{iH}}} - \frac{q_{\mathrm{L}}}{T_{\mathrm{iL}}}. \tag{6}$$

Solving now for P we obtain

$$P = q_{\mathrm{H}}\left(1 - \frac{T_{\mathrm{iL}}}{T_{\mathrm{iH}}}\right) = K(T_{\mathrm{H}} - T_{\mathrm{iH}})\left(1 - \frac{T_{\mathrm{L}}}{T_{\mathrm{iH}}}\right). \tag{7}$$

For given temperatures of the heat baths and given K, the flow of heat through the engine and the power produced by the inner Carnot engine will depend only on the operating temperatures of the Carnot engine. As T_{H} and T_{L} are fixed, the only control to influence the overall performance of the endoreversible engine is T_{iH}, and we find the power P as a function of T_{iH} only. Equation (7) alone is thus characterizing the entire endoreversible Novicov engine with Newtonian heat conduction.

The maximum power is determined by differentiation with respect to T_{iH}

$$0 = \frac{dP}{dT_{\mathrm{iH}}} = K\left(\frac{T_{\mathrm{H}}T_{\mathrm{L}}}{T_{\mathrm{iH}}^2} - 1\right), \tag{8}$$

from which we find $T_{\mathrm{iH}} = \sqrt{T_{\mathrm{H}} T_{\mathrm{L}}}$. Operating with this temperature the maximum power is

$$P_{max} = K\left(\sqrt{T_{\mathrm{H}}} - \sqrt{T_{\mathrm{L}}}\right)^2 \tag{9}$$

and the efficiency in terms of the bath temperatures is

$$\eta(P_{max}) = 1 - \frac{T_{\mathrm{iL}}}{T_{\mathrm{iH}}} = 1 - \sqrt{\frac{T_{\mathrm{L}}}{T_{\mathrm{H}}}}. \tag{10}$$

The reader should note the remarkable fact that this efficiency does not depend on the size of the heat conductance K. Also note that this efficiency is not a bound for heat engines operating not at the maximum power point.

This simple example has shown how with a relatively modest effort new and interesting results can be obtained for the performance of heat engines operating out of equilibrium.

3. Endoreversible systems – a formal description

We now generalize the ideas presented above and view an endoreversible system as a network of reversible subsystems exchanging energy. Setting up the mathematical description of an endoreversible system is quite easy, usually a number of balance equations and transport equations have to be combined. For an endoreversible system energy is not the only exchanged quantity, in each interaction between subsystems it is accompanied by another quantity, be it for instance entropy, momentum or a particle flux.

The basic building block of endoreversible systems is a thermal equilibrium system. It is described by its state variables, but as usual for equilibrium systems, one has some freedom in the choice of these state variables. In the following, X_i^α will denote the extensive thermodynamic variables of subsystem i, for instance the volume V_i, or particle number N_i, all of which are counted by α. The entropy S_i of subsystem i is a well defined state variable due to endoreversibility and belongs to the set of extensive variables. Thus the state of the subsystem is uniquely described by the set of its extensities $\{X_i^\alpha\}$. We then have

$$E_i = E_i(X_i^\alpha). \tag{11}$$

Note that (11) *defines* the properties of subsystem i, i.e. specifying E_i as a function of the X_i^α determines what the thermal behavior of that subsystem is. The energy E is not confined to be the internal energy, it can in addition include the translational kinetic energy, the rotational kinetic energy, or the potential energy in one or more external fields.

Due to endoreversibility all the standard equilibrium relations hold within a subsystem. We obtain the respective conjugate intensive variables Y_i^α from (11):

$$Y_i^\alpha = \frac{\partial E_i}{\partial X_i^\alpha} \ . \tag{12}$$

The Gibbs equation becomes

$$dE_i = \sum_\alpha Y_i^\alpha dX_i^\alpha \ , \tag{13}$$

and in each subsystem the extensive and the intensive variables are related via the equations of state (12).

Due to the Gibbs equation each influx of an extensity $J_i^\alpha = \dot{X}_i^\alpha$ into the system carries an accompanying influx of energy I_i^α [17]:

$$I_i^\alpha = Y_i^\alpha J_i^\alpha. \tag{14}$$

For instance any heat flux q is carried by an entropy flux q/T, or an angular momentum flux M (torque) carries an energy flux ωM, where ω is the angular velocity.

Each reversible subsystem i is characterized by a number of contact points (or contacts), through which the subsystem receives or discards energy. Through each contact the energy is transported by an extensity (a carrier) X_i^α, for instance entropy or volume. The contacts for the same extensity in one subsystem are numbered by r.

Each contact has three (time-dependent) functions assigned to it $(Y_i^{\alpha,r}, J_i^{\alpha,r}, I_i^{\alpha,r})$. Here $I_i^{\alpha,r}$ is the energy flux into the system, $J_i^{\alpha,r}$ is the associated flux of the carrier X_i^α and $Y_i^{\alpha,r}$ is the corresponding thermodynamic intensity for that contact. Endoreversibility guarantees that the energy and extensity influx at each contact are always related by (14).

In the following it is helpful to distinguish between two different types of subsystems, reservoirs and engines.

A **reservoir** is a thermodynamic system in equilibrium, characterized by either

a) given intensities Y_i^α. This is the case for infinite reservoirs where the influx of an extensity does not change the value of the intensity. In the introductory example both heat baths were of this type.

b) its extensities X_i^α and its energy function $E_i(X_i^\alpha)$. Then the intensities are known $Y_i^\alpha = \partial E_i / \partial X_i^\alpha$, and due to its internal equilibrium they are uniform throughout the subsystem and thus the contact intensities $Y_i^{\alpha,1} = Y_i^{\alpha,2} = \cdots \equiv Y_i^\alpha$ are equal for all r. From the balance equations for the extensities and the energy one finds

$$\dot{X}_i^\alpha = \sum_r J_i^{\alpha,r} \qquad \text{and} \qquad \dot{E}_i = \sum_{\alpha,r} I_i^{\alpha,r} = \sum_\alpha Y_i^\alpha \sum_r J_i^{\alpha,r}, \tag{15}$$

where we have assumed that the extensities are neither destroyed or produced within a subsystem.

An **engine** is a reversible subsystem, for which the contact variables are related by special balance requirements for the extensities and the energy. For an engine operating in a steady state one requires

$$0 = \sum_r J_i^{\alpha,r} \qquad \text{and} \qquad 0 = \sum_{\alpha,r} I_i^{\alpha,r} = \sum_{\alpha,r} Y_i^{\alpha,r} J_i^{\alpha,r}, \tag{16}$$

while for cyclic engines with cycle time t_{tot}

$$0 = \int_0^{t_{\text{tot}}} dt \sum_r J_i^{\alpha,r} \quad \text{and} \quad 0 = \int_0^{t_{\text{tot}}} dt \sum_{\alpha,r} I_i^{\alpha,r} = \int_0^{t_{\text{tot}}} dt \sum_{\alpha,r} Y_i^{\alpha,r} J_i^{\alpha,r} \tag{17}$$

holds. Note that for the endoreversible engine one does not need to know the equations of state for its working fluid. In the introductory example equations (5) and (6) correspond to equations (16), respectively.

The **interactions** describe, how the contacts of the subsystems exchange energy. The contact points are connected by the interactions such that each contact belongs to one specific interaction. An interaction Ω is characterized by the set of contacts which belong to it, and by the specific extensity X^α, through which the contacts exchange energy. If the interaction is reversible, only contacts for the exchanged extensity are needed. If the interaction is irreversible, entropy is produced, and then at least one additional contact is needed in which the produced entropy can be deposited.

Some of the extensities (like angular momentum) and energy are conserved quantities by nature, while others (like the particle number of chemical species) are not. In a (complete) interaction all the conserved quantities must balance to zero. We shall see later however, that often it suffices to consider only partial interactions. A specific interaction Ω can be either reversible or irreversible, and can be either of the two following cases:

a) All the contact intensities $\nu, \mu \in \Omega$ obey $Y_\nu = Y_\mu$. In the above discussed Novicov engine the interaction between the engine and the lower heat bath is of this type.
b) The interaction is defined by a transport law which gives either the flux of the extensity

$$J_\nu = J_\nu(\{Y_\omega\}, \{X_i^\alpha\}, z_m) \tag{18}$$

or the corresponding flux of the energy

$$I_\nu = I_\nu(\{Y_\omega\}, \{X_i^\alpha\}, z_m) \tag{19}$$

at each of the involved contacts as functions of the intensities, the extensities (for reservoirs) and of additional external parameters z_m, which are counted by m. These parameters are mentioned here explicitly, as they are sometimes used as 'controls' to adjust the fluxes in optimizing the performance of endoreversible systems.

In the above discussed Novicov engine the interaction of the engine to the upper heat bath is of type b. Here two entropy contacts are coupled such that the heat flux (3) is a function of the two bath temperatures, and K is an external parameter, one of the z_m. The entropy fluxes do not balance as the transport is irreversible.

In general these interactions can be much more complicated, and the form of the heat transfer law will significantly influence the behavior of an endoreversible system. Other heat transfer laws than the so-called Newton heat transfer law discussed above are the Fourier and radiative heat transfer.

For the Fourier heat transfer law the heat flux q is proportional to the difference of the inverse temperatures,

$$q = K(1/T_2 - 1/T_1) \tag{20}$$

where K is an Onsager coefficient. This form of heat transfer is often found in conjunction with linear irreversible thermodynamics, as there the difference of the inverse temperatures is the force corresponding to the heat flux. In the case of a Novikov-engine such a law will lead to an efficiency at maximum power of $\eta(P_{max}) = \eta_{\mathrm{C}}/2$.

Radiative heat transfer is typically described by the Stefan-Boltzmann law for black-body radiation, and the heat flux between two radiating bodies at temperature T_1 and T_2 is given by

$$q = K_1 T_1^4 - K_2 T_2^4\ . \tag{21}$$

The coefficients K are proportional to the Stefan-Boltzmann constant, the emittances of the two radiating bodies, and geometry factors. Solar collectors are typical applications where radiative heat transfer is involved as an interaction.

3.1. The characterization of endoreversible systems. Collecting the different elements introduced above, we find that an endoreversible system is described by its contact variables and the extensities of the reservoirs. If the system does not contain finite capacity reservoirs, then the extensities of the reservoirs can be excluded from the description. Usually some of the contact variables and extensities will be given, for instance the temperatures of some heat bath, while others remain undetermined. They are however not completely free, as all of them are related by constraints due to

- the Gibbs relation at each contact
- the balance equations in the reservoirs
- the balance equations in the engines
- the interactions.

Thus an endoreversible system is completely characterized by this set of algebraic and ordinary differential equations relating its contact variables and reservoir extensities.

3.2. The performance of endoreversible systems. We now turn to the analysis of endoreversible systems. A large number of different systems have been analyzed in the literature, using different schemes and levels of sophistication. The level of mathematical sophistication needed depends crucially on the question, whether time-dependent contact variables and subsystem extensities are present or not. If they are not present the level of mathematical sophistication reduces considerable. Time-dependent endoreversible systems on the other hand require often the use of control theory or of the calculus of variation for the discussion of their performance extremes [18, 19].

In steady state operation all contact variables are time-independent, for instance reservoirs are characterized by stationary intensities. Then all the contact variables become simple variables (For cyclic operation the situation is a little more complicated, see [15, 16]). In a geometric sense the whole endoreversible system is thus characterized by a hypersurface of all possible operating points in the multidimensional space spanned by all the contact variables.

Sometimes one is not only interested in discussing the performance of an endoreversible system as a function of its thermodynamic (contact) variables, but also as a function of certain external parameters z_m. These can be for instance the heat conduction areas in heat exchangers. Due to economic constraints the total heat exchanger inventory might be given, and the question might be how the performance is changed by different allocations. In such a case the contact variables are supplemented by these parameters to form a higher dimensional space, and the added constraints together with the thermodynamic constraints lead to a hyper-surface of possible operating and design points.

As the characterization of an endoreversible system by its hyper-surface is quite complicated the usual next step in the analysis of an endoreversible system is the study of certain performance measures [20], which are defined on the contact variables (and the external parameters) and thus on each (operating) point on the hyper-surface. In a way such

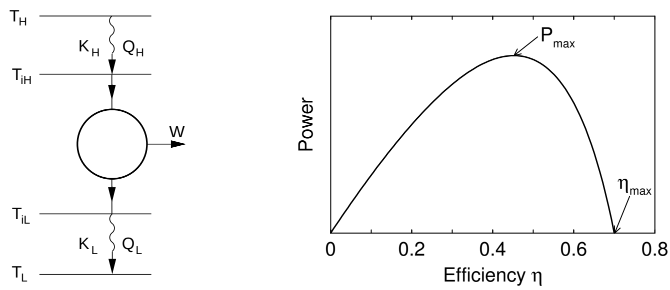

FIGURE 2. (left:) Curzon–Ahlborn model of an endoreversible heat engine with finite heat transfer to and from the heat reservoirs.
(right:) Power vs. efficiency plot for an endoreversible Curzon–Ahlborn engine with finite heat transfer.

a performance measure can be considered as a projection of the complicated hyper-surface onto one dimension.

Sometimes also two-dimensional projections are considered, where one performance measure is discussed as a function of another. Examples for performance characteristics of this sort are the power–efficiency curve [21] and the COP (coefficient of performance) vs. cooling load curve.

Often the extreme values which performance measures can achieve are used in the characterization of endoreversible systems. In our introductory example the power as one performance measure was maximized, and the efficiency of the engine at that operating point was determined.

Other measures are efficiency, the COP for refrigerating devices, and in particular the entropy production rate

$$\sigma = \sum_{i,r} J_i^{S,r}. \tag{22}$$

The concept of entropy minimization has been used extensively. In particular, Bejan has published numerous articles and books on the topic (see [22, 23], for example).

4. A cyclic operating heat engine

In this section we will bring the formal description of endoreversible thermodynamics to life by applying it to the Curzon–Ahlborn engine [8], see Figure 2. It consists of two heat baths at constant temperatures T_{H} and T_{L} and a reversible Carnot engine operating between the temperatures T_{iH} and T_{iL}.

We here consider a cyclicly operating engine. Delivering the work W per cycle, the engine absorbs the heat Q_{H} from the hot temperature reservoir during the time t_{H} and rejects the heat Q_{L} to the low temperature reservoir during the time t_{L}:

$$Q_{\mathrm{H}} = K_{\mathrm{H}} t_{\mathrm{H}} (T_{\mathrm{H}} - T_{\mathrm{iH}}) \tag{23}$$

$$Q_{\mathrm{L}} = K_{\mathrm{L}} t_{\mathrm{L}} (T_{\mathrm{iL}} - T_{\mathrm{L}}) \tag{24}$$

where $K_{\rm H}$ and $K_{\rm L}$ are the respective thermal conductances. The time spent in the isentropic branches of the Carnot cycle is considered to be negligible compared to the isotherms such that the total cycle time is the sum of the times spent in the isothermal branches: $t_{\rm tot} = t_{\rm H} + t_{\rm L}$.

The goal is now to determine the maximum work W per cycle (which also maximizes the maximum average power output as the total cycle time is fixed) and the efficiency at that operating point. We find

$$W = C T_{\rm H} \eta \frac{\eta_{\rm C} - \eta}{1-\eta} \tag{25}$$

where we introduced

$$\eta = 1 - T_{\rm iL}/T_{\rm iH} \qquad C = \left(\frac{1}{\kappa_{\rm H} t_{\rm H}} + \frac{1}{\kappa_{\rm L} t_{\rm L}}\right)^{-1} . \tag{26}$$

A plot of the power versus efficiency characteristics (25) is depicted in Figure 2.

One can obtain the maximum work point by setting both $(\partial W/\partial \eta)_{t_{\rm H}}$ and $(\partial W/\partial t_{\rm H})_\eta$ equal to zero. The condition $(\partial W/\partial t_{\rm H})_\eta = 0$ gives a relation between the branch times and the conductances:

$$t_{\rm H}/t_{\rm L} = \sqrt{K_{\rm L}/K_{\rm H}} \, . \tag{27}$$

The condition $(\partial W/\partial \eta)_{t_h} = 0$ leads to the Curzon–Ahlborn efficiency

$$\eta(P_{\rm max}) = \eta_{\rm CA} = 1 - \sqrt{T_{\rm L}/T_{\rm H}} \tag{28}$$

at maximum work, which is obtained by inserting (28) and (27) into (25)

$$W_{\rm max} = t_{\rm tot} \frac{K_{\rm H} K_{\rm L}}{\left(\sqrt{K_{\rm H}} + \sqrt{K_{\rm L}}\right)^2} \left(\sqrt{T_{\rm H}} - \sqrt{T_{\rm L}}\right)^2 . \tag{29}$$

Note that $W_{\rm max}$ still depends on the conductances $K_{\rm H}$ and $K_{\rm L}$.

The Curzon–Ahlborn efficiency is much closer to observed efficiencies than the corresponding Carnot efficiencies [8, 24]. Nonetheless a careful analysis of the dissipative processes as well as of the optimization goals in real engines remain important [25].

Heat engines with heat leaks. We conclude this section by mentioning refinements of these models by introducing an internal heat bypass or heat leak. Such systems were first analyzed by Bejan [26] and later by Gordon and Huleihil [25]; they showed that the model with additional heat bypass gives an even more realistic behavior especially for slow operation. Figure 3 shows an extended Curzon–Ahlborn model with heat leak. Then if the endoreversible engine operates fast, the internal temperature difference $T_{\rm iH} - T_{\rm iL}$ becomes small and the efficiency of the endoreversible engine degrades. If on the other hand the engine operates slow, heat is lost through the heat leak causing a decrease in efficiency. Between this extremes, there are operation modes with internal temperatures $T_{\rm iH}$ and $T_{\rm iL}$ for which either the efficiency or the power output of the engine is maximized. Later work [27] showed that a Novikov engine with heat leak can even quantitatively model real combustion engines for properly chosen parameters.

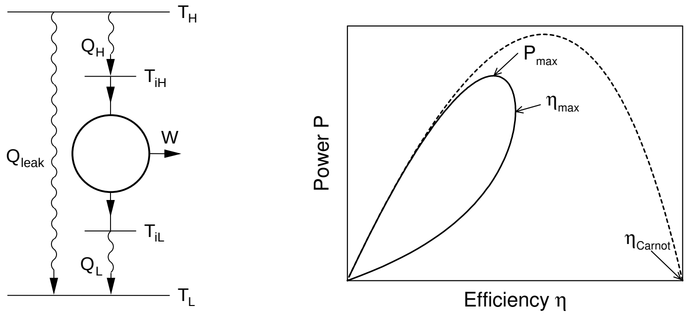

FIGURE 3. Schematic diagram of an endoreversible engine with a by-pass heat leak *(left)* and plots of average power (work per cycle time) P versus efficiency η, for an engine with a heat leak *(right)*. The dashed line corresponds to a model without heat leak. The solid line shows the loop-type behavior of a model with heat leak. Both, power *and* efficiency vanish in the thermal short-circuit limit of very fast operation *and* in the limit of very slow engine operation. The maximum power and the maximum efficiency point are relatively close together (see [25]).

5. Mathematical Tools for optimal process paths

Determining the optimal path for a thermodynamic process requires tools more elaborate than calculus. Two of these tools, the Euler-Lagrange formalism and optimal control theory, have been used widely. Here we concentrate on optimal control theory.

5.1. Optimal control theory. Optimal Control Theory evolved in the middle of the 20th century. It considers a dynamic system whose state is described by its vector of state variables $x(t) = (x_1(t), \dots, x_n(t))$, which can change in time according to

$$\dot{x} = f(x, u, t). \tag{30}$$

Here $u(t) = (u_1(t), \dots, u_r(t))$ is the vector of control variables. The set of control variables includes both the controllable state variables and external controls. Such an external control could for instance be a variable which switches on or off the coupling to a heat bath. The path $x(t)$ of the system in state space can be externally influenced by the choice of the values of the control variables out of the set of allowed controls U and is fully determined by an initial state $x^0 = x(0)$ and the controls $u(t)$.

The problem of optimal control is then to find the controls for which a given functional J becomes extremal, i.e.

$$J = \int_0^\tau f_0(x, u, t)\mathrm{d}t \to \max_{u(t)}. \tag{31}$$

First one has to form the Hamiltonian for this problem introducing the time-dependent adjoint (or co-state) variables λ_i, one for each state variable:

$$H = f_0 + \sum_{i=1}^{n} \lambda_i f_i, \tag{32}$$

where f_i are the components of the vector f in (30). The optimal controls $u^*(x, \lambda, t)$ are found by maximizing the Hamiltonian for fixed x, λ, and t (Pontryagin's maximum principle):

$$H(x, \lambda, u^*, t) \geq H(x, \lambda, u, t) \qquad \forall u \in U. \tag{33}$$

Note that the resulting controls can be discontinuous. The optimal controls are then inserted back into the canonical equations of motion. This results in a set of closed coupled differential equations with boundary conditions for x:

$$\dot{\lambda}_i = -\frac{\partial}{\partial x_i} H(x, \lambda, u, t)\Big|_{u=u^*(x,\lambda,t)} \tag{34}$$

$$\dot{x}_i = \frac{\partial}{\partial \lambda_i} H(x, \lambda, u, t)\Big|_{u=u^*(x,\lambda,t)}. \tag{35}$$

If the final value of a state variable is not given, the corresponding adjoint variable is required to have the final value equal to zero (transversality condition).

Depending on the problem the optimal path may consist of several arcs which need to be connected in the so-called switching problem. Across the switchings controls can have jumps. Sometimes the solution consists of an arc which represents an interior optimum which is bracketed by arcs representing boundary solutions of the controls or even jumps in state variables. Such solutions have been dubbed 'turnpikes' [28].

5.2. Path optimization of a Novikov engine. A simple example for an endoreversible system is the Novikov engine [3] introduced above. The cycle is characterized by two isotherms with temperatures T_{iH} and T_{iL} and two isentrops with entropies S_1 and S_2.

We now turn this model into a dynamical one: What is the optimal cycle process if we drop the assumption that the working fluid undergoes a Carnot cycle? To be more specific we still assume that the working fluid is coupled to the low temperature bath with T_{L} without any resistance. Thus $T_{\mathrm{iL}} = T_{\mathrm{L}}$ and the system undergoes an isothermal transformation from state 2 with entropy S_2 to state 1 with entropy S_1 (Fig. 4(a)). The working fluid is still coupled to the high temperature bath with T_{H} through the finite heat conductance K i.e. $q_{\mathrm{H}} = K(T_{\mathrm{H}} - T_{\mathrm{iH}})$. The time dependencies of $T(t)$ and $S(t)$ are in general unknown (Fig. 4(b)).

The objective now is to determine that cycle which maximizes the work output in a given cycle time τ, i.e. the power. The strategy is to determine the optimal path of the upper part of the cycle for every choice of the parameters S_1 and S_2 and then to perform a simple parameter optimization with respect to S_1 and S_2.

To set up the control problem we choose the entropy of the working fluid S as the state variable which evolves according to

$$\dot{S} = \frac{K(T_{\mathrm{H}} - T)}{T} \tag{36}$$

with the boundary conditions $S(0) = S_1$ and $S(\tau) = S_2$.

The temperature of the working fluid is taken as the control variable, which can be adjusted by appropriate volume changes.

As the heat transport to the low temperature bath is unlimited we assume that it happens instantaneously. Thus the full cycle time τ can be assigned to the branch which is subject

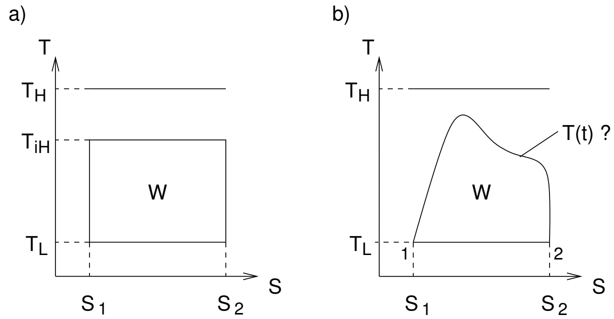

FIGURE 4. (a) The T-S diagram of a Novikov engine with a Carnot cycle taken as the reversible process. (b) In general the optimal path from state 1 to state 2 is unknown and can be found using control theory.

to optimization. The work output, i.e. the area enclosed by the T-S curve, is then given by

$$W = \int_0^\tau T\dot{S}\mathrm{d}t - T_\mathrm{L}(S_2 - S_1) = \int_0^\tau K(T_\mathrm{H} - T)\,\mathrm{d}t - T_\mathrm{L}\Delta S \tag{37}$$

with $\Delta S = S_2 - S_1$. As $T_\mathrm{L}\Delta S$ is a constant it plays no role for the path optimization, i.e. it is sufficient to maximize the integral.

Thus the Hamiltonian of the problem is

$$H = K(T_\mathrm{H} - T) + \lambda_S \frac{K(T_\mathrm{H} - T)}{T} \tag{38}$$

and the canonical equation is written as

$$\dot{\lambda}_S = -\frac{\partial H}{\partial S} = 0. \tag{39}$$

The optimal temperature path (control) is found by maximizing the Hamiltonian:

$$\frac{\partial H}{\partial T} = -K - \frac{\lambda_S K T_\mathrm{H}}{T^2} = 0 \quad \Rightarrow \quad T = \sqrt{-\lambda_S T_\mathrm{H}}. \tag{40}$$

This result has to be inserted into the set of differential equations (36), (39). λ_S turns out to be a constant. Its size is chosen such that the boundary conditions for the entropy are fulfilled. We find

$$S(t) = \frac{S_2 - S_1}{\tau} t + S_1, \qquad T(t) = \frac{K\tau T_\mathrm{H}}{\Delta S + K\tau}, \tag{41}$$

where the last expression is obtained from (36).

The temperature is a constant not equal to T_L. In order to close the cycle with the low temperature branch we need two instantaneous adiabatic jumps which connect the two isothermal branches. We see that the optimal cycle which was determined by control theory turns out to be the Carnot cycle with two adiabats and two isotherms.

There are still two free parameters S_1 and S_2 which we need to adjust to maximize the power output further. We insert the expression (41) for the optimal temperature into (37):

$$W = \frac{T_{\mathrm{H}}}{(K\tau)^{-1} + \Delta S^{-1}} - T_{\mathrm{L}} \Delta S, \tag{42}$$

and see that the work output only depends on ΔS. So we set the derivative with respect to ΔS zero and find the optimal value in terms of the given engine parameters $T_{\mathrm{H}}, T_{\mathrm{L}}, K$ and τ:

$$\Delta S^{\mathrm{opt}} = K\tau \left(\sqrt{\frac{T_{\mathrm{H}}}{T_{\mathrm{L}}}} - 1 \right). \tag{43}$$

Inserting the optimal entropy difference into (41) finally yields

$$T^{\mathrm{opt}} = \sqrt{T_{\mathrm{H}} T_{\mathrm{L}}}, \tag{44}$$

a result which leads to the well-known Curzon-Ahlborn efficiency.

6. Optimal Paths for Internal Combustion Engines

Finite-time thermodynamics started out as a reaction to the oil crisis in the early seventies. So it was very natural that after the necessary tools had become available the attention focused on the application of finite-time thermodynamics to combustion engines. Of course the idea was not to repeat nearly 100 years of careful engineering and optimization, the idea was to abstract the engines enough to make them treatable and yet to include at the same time all major loss terms [29] so that the results of the analysis would be useful in guiding which loss terms could be most easily reduced. The focus was not on optimizing the technical realization of the engines but on optimizing the thermodynamic process itself and on finding its inherent limits when performed in a finite time. The path optimization shows which loss term can be most easily reduced, and how close real engines approach these performance bounds. Dynamic endoreversible models of internal combustion engines, especially Diesel and Otto engines, have been first investigated by Mozurkevich and Berry [30], by Hoffmann, Watowich and Berry [31], and later extended by Blaudeck and Burzler [32, 33].

6.1. The Otto engine. Mozurkevich and Berry [30] investigated an internal combustion engine based on an Otto cycle. Their objective was to find the optimal piston path for which the output of work is maximized for a fixed cycle time and a given amount of fuel. The model is based on a four stroke Otto cycle with several sources of irreversibility such as piston friction and heat leak. The models employed to describe the losses were chosen such that they adequately model the qualitative behavior and the total magnitude of these losses.

Losses due to friction are approximated by a friction force linear in the piston velocity $F = \alpha v$ corresponding to a well-lubricated system. The value of α during the power stroke is assumed to be twice as large as during the other strokes due to the greater pressure. The pressure drop due to the viscosity of the gas as it flows through the intake valve is also proportional to the velocity and was thus included in the respective friction term. The pressure drop during the exhaust stroke was neglected.

Heat conduction through the cylinder walls constitutes a major loss term. The thermal conductance through the cylinder wall is taken to be proportional to the surface area inside the cylinder and the difference between the temperature of the working fluid T and that of the cylinder walls T_{ex}, which is assumed to be constant. For heat conduction coefficient κ and cylinder diameter b the rate of heat leak $q(x,t)$ at piston position x is given by:

$$q(x,t) = \kappa\pi b(0.5b + x)(T - T_{\mathrm{ex}}). \tag{45}$$

Mozurkevich and Berry [30] neglected the heat leak during the non-power strokes since the temperature difference between the working fluid and the cylinder wall is much lower during the non-power strokes than during the power stroke.

To find the optimal cycle for a given total cycle time one has to calculate the optimal path for each stroke as a function of time needed for this stroke. Then one has to determine the optimal allocation of the time to each stroke using the work output as objective function.

The optimal path on each stroke is determined by optimal control theory.

Mozurkevich and Berry obtained a number of highly interesting results. The increase of effectiveness is calculated for several sets of parameters and different values for the constraints of piston acceleration. It turns out that compared to a conventionally operated piston the magnitude of achievable increase of effectiveness due to path optimization is about 10%. Especially for those parameter sets with large heat leaks the optimal movement requires a fast expansion to avoid heat losses during the expansion. For details see ref. [30].

6.2. The Diesel engine. The main difference of the Diesel combustion engine to the Otto cycle is due to the finite combustion rate of the fuel which progresses also during the power stroke. This leads to a higher in principle efficiency of the Diesel engine. So it is very interesting to analyze a Diesel engine model to determine performance limits for this type of process. Hoffmann, Watowich, and Berry [31] performed such an analysis. The finite rate combustion process is approximated by the following time-dependent function, which describes the extent of the reaction

$$\xi(t) = F + (1-F)(1 - \exp(-t/t_b)). \tag{46}$$

The explosion fraction F is the part of fuel consumed in an initial instantaneous burn and t_{b} gives the time during which most of the combustion occurs. This yields a heating function

$$f(t) = Q_{\mathrm{c}}\dot{\xi}(t), \tag{47}$$

where Q_{c} is the heat of combustion per molar fuel-air mixture charge.

The mole number N and the heat capacity C are also assumed to be influenced by the extent of the combustion reaction. Heat losses and frictional losses were modelled as in the Otto engine. The temperature and piston position are state variables and subject to the constraints

$$\begin{aligned} \dot{T} &= \frac{f(t) - NRTv/x - q - \dot{\xi}[C(N_{\mathrm{f}} - N_{\mathrm{i}}) + N(C_{\mathrm{f}} - C_{\mathrm{i}})]}{NC}, && (48)\\ \dot{x} &= v. && (49) \end{aligned}$$

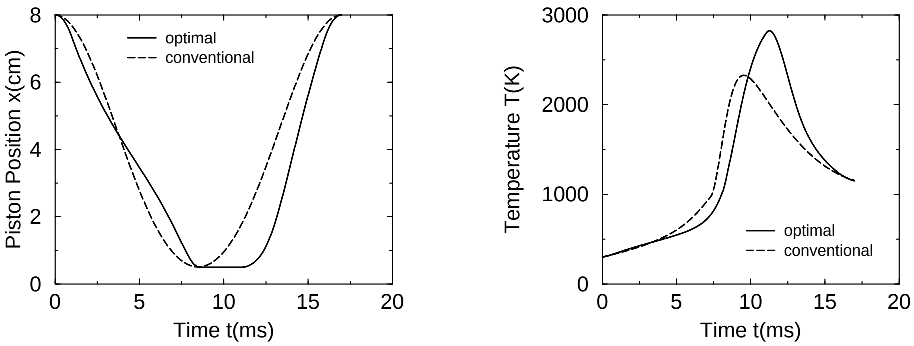

FIGURE 5. Optimal and conventional piston path of the compression and power strokes in a Diesel engine. Note the initial stand-still of the piston during the power stroke *(left)*. Temperature of the working fluid for optimal and conventional path *(right)*.

These equations lead to an optimal control problem which has to be solved numerically. The optimal piston motion showed a very surprising result: The piston should not move at all for the first part of the power stroke. This behavior seems highly wasteful, as it increases the frictional losses due to the higher velocity needed in the remaining time for the power stroke. However it turned out that due to the piston remaining fixed the temperature of the working fluid can increase higher this way. This in turn means that the available heat energy of the fuel is provided to the system with higher exergy or availability content. This shows eventually up in a higher work and power output. Again the engine efficiency was about 10% higher than for a conventionally operated piston. Later Burzler, Blaudeck and Hoffmann [33] moved the model even closer to real engines by investigating different heat loss models.

As an alternative to control theory a Monte-Carlo method can be used [32] as effectively. The results for the optimal operation of the compression and power stroke of a Diesel engine are shown in Fig. 5, where both strokes were optimized together using the Monte-Carlo method. Again the optimal power stroke begins with a short delay, where the piston remains at its extremal position. So this interesting behavior, which increases losses due to heat leak and friction but increases the temperature of the working fluid and the maximum availability of the system, remains unchanged.

7. Conclusion

In this review we considered endoreversible systems ranging from general thermodynamic cycles with different model assumptions to semi-realistic combustion engine models. Various examples showed how different techniques are applied to determine performance optima and the corresponding optimal process path for such systems. These investigations show how a specific thermodynamic system is optimally driven, how model assumptions influence the optimal path, and provide guidelines to improve real thermodynamic processes.

Endoreversible thermodynamics in our view is the successful attempt to include irreversibilities and dissipative processes into the description of thermodynamic processes, while at the same time preserving the advantages of classical reversible thermodynamics. The central idea is to think of a system as a network of subsystems – each undergoing only reversible processes – which exchange energy. All irreversibilities occur only in the interactions between the subsystems. Treating systems in this way one gets one step closer to a realistic description of real dissipative processes.

In this review we presented the general framework for the endoreversible description of thermodynamic systems undergoing irreversible processes. Depending on the desired accuracy of the description a system can be separated into a larger or smaller number of subsystems. This way the irreversibilities of the energy exchange between the parts of the system can be taken into account. The discussion focused equally on obtaining a proper mathematical theory and on the characterization of the systems performance.

Our aim was not so much to elaborate a large number of different energy transformation devices, our aim was a presentation from a systematic point of view. As a good starting point for further reading of this subject and related topics we recommend references [15, 16, 34–36].

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