

## SOME RESULTS ON THE DYNAMICS GENERATED BY THE BAZYKIN MODEL

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ABSTRACT. A predator-prey model formerly proposed by A. Bazykin *et al.* [*Bifurcation diagrams of planar dynamical systems* (1985)] is analyzed in the case when two of the four parameters are kept fixed. Dynamics and bifurcation results are deduced by using the methods developed by D. K. Arrowsmith and C. M. Place [*Ordinary differential equations* (1982)], S.-N. Chow *et al.* [*Normal forms and bifurcation of planar fields* (1994)], Y. A. Kuznetsov [*Elements of applied bifurcation theory* (1998)], and A. Georgescu [*Dynamic bifurcation diagrams for some models in economics and biology* (2004)]. The global dynamic bifurcation diagram is constructed and graphically represented. The biological interpretation is presented, too.

### 1. Introduction

This paper deals with a particular family of planar vector fields which models the dynamics of two populations. Predator-prey models govern many phenomena in population dynamics, immunology, medicine etc. We assume that there are only two competing species: one species (predator) feeds on another species (prey), which in turn feeds on other things. The model is a variation of the Lotka-Volterra system, a particular case of the Bazykin model, [1]

$$(1) \quad \begin{cases} \dot{x} &= x(1 + \alpha x - \varepsilon x - y - \alpha \varepsilon x^2), \\ \dot{y} &= y(-1 - \alpha x + x), \end{cases}$$

where  $x$  and  $y$  represent the population numbers of the prey and the predator, respectively,  $\varepsilon$  is the prey competition rate and  $\alpha$  determines the saturation of the predator. The initial Bazykin model is [2]

$$(2) \quad \begin{cases} \dot{x} &= x - \frac{xy}{1 + \alpha x} - \varepsilon x^2, \\ \dot{y} &= -\gamma y + \frac{xy}{1 + \alpha x} - \delta y^2, \end{cases}$$

where  $x$  and  $y$  represent the population numbers of the prey and the predator, respectively, and  $\alpha$ ,  $\varepsilon$ ,  $\gamma$  and  $\delta$  are nonnegative parameters describing the behaviour of isolated populations and their interaction (namely  $\alpha$  determines saturation of predator,  $\varepsilon$ ,  $\delta$  represent the prey and predator competition rates, and  $\gamma$  represents the predator natural mortality rate).

The presence of the four parameters is the source of a very rich dynamics generated by (2) and of its qualitative changes as the vector parameter  $(\alpha, \varepsilon, \gamma, \delta)$  crosses some manifolds in the parameter space. These changes correspond to qualitative modification of the populations behaviour and to topological changes of the phase portraits of the associated dynamical system and are called the dynamical bifurcations. The geometric representation of all possible bifurcations are given by the global dynamic bifurcation diagram consisting of the parameter portrait (formed by the quoted manifolds) and the corresponding phase portraits.

The model (2) was studied by Bazykin in 1985 [1] and, so, a lot of bifurcation results were obtained. In a series of papers we treat the bifurcation from a new perspective [3]. So, for example, in [4] it is numerically treated the general case with four parameters for different values of the parameters, with the CONTENT program. Then it is taken  $\beta = 0.01$  and  $\gamma = 1$  and is studied the parametric portrait in the  $(\alpha, \delta)$  plane. There are found Hopf, Bogdanov-Takens, cusp bifurcation points. In [2] it is considered the case  $\varepsilon \ll 1$  and  $\gamma = 1$ . It is studied the parametric portrait in the  $(\alpha, \delta)$  plane and there are founded fold, saddle, saddle-node, Hopf, Bautin, Bogdanov-Takens bifurcation points. In 1974 Bazykin proposed the more complex model

$$(3) \quad \begin{cases} \dot{x} &= xa - \frac{bxy}{1 + \alpha x} - \varepsilon x^2, \\ \dot{y} &= -cy + \frac{dxy}{1 + \alpha x}, \end{cases}$$

where  $a, b, c, d, \alpha, \varepsilon$  are positive parameters. In [5] a numerical approach ( $a = 0.6, b = 0.3, c = 0.4, d = 0.5$ ) is carried out and a study in the  $(\varepsilon, \alpha)$  plane is made by means of the XPPAUT program. The Hopf, transcritical and saddle bifurcation points are found.

Because in [2] there are four parameters, the study of the model is very complicated and not yet completely made. So, we make a section in the parametric space (a limit situation), keeping fixed two parameters (namely  $\gamma = 1$  [2] and, for simplicity,  $\delta = 0$ ). Our aim is to study this limit case, and to determine the possible situations which can appear. All of them are studied from the mathematical point of view. A genuine biological interpretation is not appropriate because this case is not admissible. However, a lot of phenomena occurring in biologically admissible cases are consequences of the mathematical phenomena (and especially bifurcations) associated with limit cases. This motivate our study. We present such a limit case which proves to be very rich in bifurcation phenomena.

For the beginning, we start with the study of the equilibrium points. We present here the proof only for a nonhyperbolic equilibrium point (namely the saddle-node), for the others (namely the Hopf singularities), the proof is presented in other paper [6]. Then we represent the global dynamic bifurcation diagram used the WINPP program. Finally, we try to give a biological interpretation, mostly based on our (namely mathematical) viewpoint.

## 2. The Bazykin predator-prey model

We deal with a particular case of the model consisting of a Cauchy problem  $x(0) = x_0, y(0) = y_0$ , for the system of ordinary differential equations (sode) (2).

In this paper we study only the case  $\gamma = 1$ ,  $\delta = 0$ , when (2) becomes

$$(4) \quad \begin{cases} \dot{x} &= x - \frac{xy}{1 + \alpha x} - \varepsilon x^2, \\ \dot{y} &= -y + \frac{xy}{1 + \alpha x}, \end{cases}$$

or, equivalently,

$$(5) \quad \begin{cases} \dot{x}(1 + \alpha x) &= x(1 + \alpha x) - xy - \varepsilon x^2(1 + \alpha x), \\ \dot{y}(1 + \alpha x) &= -y(1 + \alpha x) + xy. \end{cases}$$

Introducing the new time  $\tau$  through the relation  $dt = (1 + \alpha x)d\tau$ , (5) becomes

$$(6) \quad \begin{cases} \dot{x} &= x(1 + \alpha x - \varepsilon x - y - \alpha \varepsilon x^2), \\ \dot{y} &= y(-1 - \alpha x + x), \end{cases}$$

where the dot over quantities stands for the differentiation with respect to  $\tau$ . This is the sode we are concerned with herein.

Due to physical reasons, the phase space must be the first quadrant (without axes of coordinates). However, as we already said, for mathematical (namely bifurcation) reasons we consider, in addition, the origin, the half-axes and the other quadrants.

### 3. The equilibrium points

By convention, we say that an equilibrium exists if its coordinates are finite and positive. Therefore, this is a biological, not a mathematical existence.

**Case  $\alpha = \varepsilon = 0$ .** In this case (6) takes the form of the classical Lotka-Volterra model

$$(7) \quad \begin{cases} \dot{x} &= x(1 - y), \\ \dot{y} &= y(-1 + x). \end{cases}$$

The equilibrium points are the saddle  $O(0, 0)$  and the center  $A(1, 1)$ .

We recall that the attractivity properties of an equilibrium point  $(x^*, y^*)$  is determined by the real part of the eigenvalues of the matrix defining the linearized sode about this point.

**Case  $\alpha = 0$ ,  $\varepsilon \neq 0$ .** In this case (6) becomes

$$(8) \quad \begin{cases} \dot{x} &= x(1 - \varepsilon x - y), \\ \dot{y} &= y(-1 + x). \end{cases}$$

and has the equilibrium points  $O(0, 0)$ ,  $E(1/\varepsilon, 0)$  and  $A(1, 1 - \varepsilon)$ . The point  $O$  is a saddle;  $E$  is a saddle for  $1 - \varepsilon > 0$ , a saddle-node for  $\varepsilon = 1$  and an attractive node for  $1 - \varepsilon < 0$ . The point  $A$  is an attractive focus for  $\varepsilon \in (0, 2\sqrt{2} - 2)$ , a sink for  $\varepsilon = 2\sqrt{2} - 2$ , and an attractive node for  $\varepsilon \in (2\sqrt{2} - 2, 1)$ . For  $\varepsilon = 1$ , at  $C_2$ , the point  $A$  collides with  $E$  and it becomes a saddle-node. For  $\varepsilon > 1$ ,  $A$  is a saddle.

**Case  $\alpha \neq 0$ ,  $\varepsilon = 0$ .** In this case (6) becomes

$$(9) \quad \begin{cases} \dot{x} &= x(1 + \alpha x - y), \\ \dot{y} &= y(-1 - \alpha x + x). \end{cases}$$

The equilibrium points are  $O(0, 0)$ ,  $B(-1/\alpha, 0)$  and  $A((1 - \alpha)^{-1}, (1 - \alpha)^{-1})$  for  $\alpha \neq 1$ . The point  $O$  is a saddle,  $B$  is an attractive node for  $\alpha \neq 1$  and a degenerated attractive

node for  $\alpha = 1$ , while  $A$  is a repulsive focus if  $\alpha \in (0, 2\sqrt{2}-2)$ , a source if  $\alpha = 2\sqrt{2}-2$ , a repulsive node if  $\alpha \in (2\sqrt{2}-2, 1)$ , a saddle if  $\alpha > 1$ , it does not exist mathematically for  $\alpha = 1$ , while for  $\alpha > 1$  it exists mathematically, but not biologically. Hence, according to our convention,  $A$  disappears at  $C_3(1, 0)$ , for  $\alpha = 1$ . This is why, at  $\alpha = 1$ ,  $A$  is not a double zero equilibrium as expected. As a consequence, at  $C_3$  the curves  $H_{\mathbb{R}}$ ,  $H_{\mathbb{C}}$  and  $S$  are not tangent (these curves are defined in Section 4).

**Case  $\alpha \neq 0$ ,  $\varepsilon \neq 0$ .** The system (6) can have the following equilibrium points:  $O(0, 0)$ ,  $E(1/\varepsilon, 0)$ ,  $B(-1/\alpha, 0)$  and  $A((1-\alpha)^{-1}, (1-\alpha-\varepsilon)(1-\alpha)^{-2})$  for  $1-\alpha \neq 0$ . If  $1-\alpha = 0$  the point  $A$  disappears. The number and the multiplicity of the equilibrium points depend on the values of the parameters  $\alpha$  and  $\varepsilon$ .

The attractivity of an equilibrium point  $(x^*, y^*)$  of (6) is determined by the eigenvalues of the matrix

$$(10) \quad \mathbf{A} = \begin{pmatrix} 1 + 2(\alpha - \varepsilon)x - y - 3\alpha\varepsilon x^2 & -x \\ (1 - \alpha)y & -1 + (1 - \alpha)x \end{pmatrix} \Big|_{(x^*, y^*)}.$$

For the equilibrium point  $O$ ,  $\mathbf{A}$  becomes  $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , which has the eigenvalues  $\lambda_1 = 1 > 0$ ,  $\lambda_2 = -1 < 0$ . Thus  $O$  is a saddle.

For the equilibrium point  $E$ , the matrix  $\mathbf{A}$  takes the form  $\mathbf{A} = \begin{pmatrix} (-\varepsilon - \alpha)/\varepsilon & -1/\varepsilon \\ 0 & (1 - \alpha - \varepsilon)/\varepsilon \end{pmatrix}$ , and has the eigenvalues  $\lambda_1 = -1 < 0$ ,  $\lambda_2 = (1 - \alpha - \varepsilon)/\varepsilon$ . Therefore, if  $\alpha \neq 1$  and  $\varepsilon \neq 1$ ,  $E$  is a saddle for  $1 - \alpha - \varepsilon > 0$ , a saddle-node for  $1 - \alpha - \varepsilon = 0$  and an attractive node for  $1 - \alpha - \varepsilon < 0$ . For  $\alpha = 1$   $E$  is an attractive node.

For the equilibrium point  $B$ , the matrix  $\mathbf{A}$  takes the form  $\mathbf{A} = \begin{pmatrix} (-\alpha - \varepsilon)/\alpha & 1/\alpha \\ 0 & -1/\alpha \end{pmatrix}$ , which has the eigenvalues  $\lambda_1 = (-\alpha - \varepsilon)/\alpha < 0$  and  $\lambda_2 = -1/\alpha < 0$ . Thus  $B$  is an attractive node.

For the equilibrium point  $A$ , the matrix  $\mathbf{A}$  takes the form

$$(11) \quad \mathbf{A} = \begin{pmatrix} (-\alpha^2 + \alpha - \alpha\varepsilon - \varepsilon)/(1 - \alpha)^2 & -1/(1 - \alpha) \\ (1 - \alpha - \varepsilon)/(1 - \alpha) & 0 \end{pmatrix}.$$

The eigenvalues of (11) are the roots of the characteristic equation

$$(12) \quad \lambda^2 - \text{tr } \mathbf{A} \lambda + \det \mathbf{A} = 0,$$

with  $\det \mathbf{A} = (1 - \alpha - \varepsilon)/(1 - \alpha)^2$ ,  $\text{tr } \mathbf{A} = (-\alpha^2 + \alpha - \alpha\varepsilon - \varepsilon)/(1 - \alpha)^2$ , and  $\Delta = ((\alpha^2 + 2\alpha + 1)\varepsilon^2 + (2\alpha^3 + 4\alpha^2 + 4 - 10\alpha)\varepsilon + \alpha^4 + 2\alpha^3 - 11\alpha^2 + 12\alpha - 4)/(\alpha - 1)^4$ . Therefore, if  $1 - \alpha - \varepsilon > 0$   $A$  is an attractive (repulsive) focus or an attractive (repulsive) node, depending on the sign of  $\text{tr } \mathbf{A}$  and  $\Delta$ , a saddle if  $1 - \alpha - \varepsilon < 0$  and a saddle-node if  $1 - \alpha - \varepsilon = 0$ . In this last case the point  $A$  coincides with the point  $E$ .

Owing to the Hartman-Grobman theorem, we are interested only in nonhyperbolic equilibria: the center  $A(1, 1)$  for  $\alpha = \varepsilon = 0$ ; the saddle-node  $A(1/\varepsilon, 0)$  for  $0 \leq \alpha < 1$ ,  $\varepsilon = 1 - \alpha$  and the Hopf singularity  $A((1-\alpha)^{-1}, (1-\alpha-\varepsilon)(1-\alpha)^{-2})$  for  $-\alpha^2 + \alpha - \alpha\varepsilon - \varepsilon = 0$ . Since the center  $A(1, 1)$  was studied very much for the Lotka-Volterra models, we have to investigate only the other two singularities. In order to see whether  $A$  is a degenerated

or a nondegenerated singularity we have to derive the normal form of (6) at  $A$  [7]. We found that the both singularities are nondegenerated. In the following we will study only the saddle-node singularity, while the Hopf singularity will be study elsewhere.

**Remark 1.** *The axes of coordinates are separatrices of dynamics.*

#### 4. The normal form for the saddle-node

Consider the case  $0 \leq \alpha < 1$ ,  $1 - \alpha - \varepsilon = 0$ ,  $\varepsilon \neq 0$ , when the equilibrium  $A$  coincides with  $E(1/\varepsilon, 0)$ , therefore  $E$  is a double equilibrium point. First, we translate the point  $E$  at the origin with the aid of the change  $u_1 = x - 1/\varepsilon$ ,  $u_2 = y$ . Let  $\mathbf{u} = (u_1, u_2)^T$ . Then, in  $\mathbf{u}$ , (6) reads

$$(13) \quad \begin{cases} \dot{u}_1 \varepsilon &= -u_1 - u_2 + \varepsilon(\varepsilon - 2)u_1^2 - \varepsilon u_1 u_2 - \varepsilon^2(1 - \varepsilon)u_1^3, \\ \dot{u}_2 \varepsilon &= \varepsilon^2 u_1 u_2. \end{cases}$$

Introducing a new time  $\nu = \tau/\varepsilon$ , and redenoting  $\cdot \equiv \frac{d}{d\nu}$ , (13) becomes

$$(14) \quad \begin{cases} \dot{u}_1 &= -u_1 - u_2 + \varepsilon(\varepsilon - 2)u_1^2 - \varepsilon u_1 u_2 - \varepsilon^2(1 - \varepsilon)u_1^3, \\ \dot{u}_2 &= \varepsilon^2 u_1 u_2. \end{cases}$$

The eigenvalues of the matrix defining the linear terms in (14) are  $\lambda_1 = -1$ ,  $\lambda_2 = 0$  and the corresponding eigenvectors read  $\mathbf{u}_{\lambda_1} = (1, 0)^T$  and  $\mathbf{u}_{\lambda_2} = (1, -1)^T$ . Thus, with the change of the coordinates  $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ , (13) asumes the form

$$(15) \quad \begin{cases} \dot{v}_1 &= -v_1 + \varepsilon(\varepsilon - 2)v_1^2 + \varepsilon(\varepsilon - 3)v_1 v_2 - \varepsilon v_2^2 + O(\mathbf{v}^3), \\ \dot{v}_2 &= \varepsilon^2 v_1 v_2 + \varepsilon^2 v_2^2, \end{cases}$$

involving a diagonal matrix of the linear terms. In order to reduce the second-order nonresonant terms in (14) we determine the transformation  $\mathbf{v} = \mathbf{n} + \mathbf{h}(\mathbf{n})$ , where  $\mathbf{v} = (v_1, v_2)^T$  and  $\mathbf{n} = (n_1, n_2)^T$ , suggested by the Table 1

$m_1$	$m_2$	$X_{\mathbf{m},1}$	$X_{\mathbf{m},2}$	$\Lambda_{\mathbf{m},1}$	$\Lambda_{\mathbf{m},2}$	$h_{\mathbf{m},1}$	$h_{\mathbf{m},2}$
2	0	$\varepsilon(\varepsilon - 2)$	0	-1	-2	$-\varepsilon(\varepsilon - 2)$	0
1	1	$\varepsilon(\varepsilon - 3)$	$\varepsilon^2$	0	-1	-	$-\varepsilon^2$
0	2	$-\varepsilon$	$\varepsilon^2$	1	0	$-\varepsilon$	-

Table 1.

where  $\Lambda_{\mathbf{m},1}$ ,  $\Lambda_{\mathbf{m},2}$  are the eigenvalues of the associated Lie operator, and  $X_{\mathbf{m}}$  is the second-order vector polynomial in (14). We find the transformation

$$\begin{cases} v_1 &= n_1 - \varepsilon(\varepsilon - 2)n_1^2 - \varepsilon n_2^2, \\ v_2 &= n_2 - \varepsilon^2 n_1 n_2, \end{cases}$$

carrying (14) into

$$(16) \quad \begin{cases} \dot{n}_1 &= -n_1 + \varepsilon(\varepsilon - 3)n_1 n_2 + O(\mathbf{n}^3), \\ \dot{n}_2 &= \varepsilon^2 n_2^2 + O(\mathbf{n}^3). \end{cases}$$

This is the normal form of (13). In [7] it is shown that the equilibrium point  $E(1/\varepsilon, 0)$  of the dynamical system generated by a sode like (16) is a nondegenerated saddle-node.

## 5. The global dynamic bifurcation diagram

The discussion in Section 3 shows that in the parameter space  $(\alpha, \varepsilon)$  the strata are determined by: the curve  $S$  (zone 17), corresponding to saddle-nodes; the curve  $H_{\mathbb{C}}$  (zone 13), corresponding to linear centers and, so, possibly to Hopf bifurcation points, the curve  $D$  (zones 10 and 15), corresponding to double eigenvalues, and the points  $C_1(0, 2\sqrt{2} - 2)$ ,  $C_2(0, 1)$ ,  $C_3(1, 0)$ ,  $C_4(2\sqrt{2} - 2, 0)$ . Recall that the curves  $H$  (defined by  $\text{tr}\mathbf{A} = 0$ , i.e. the sum of the eigenvalues is null) and  $S$  are common to all equilibria. In addition, all  $H_{\mathbb{C}}$  (one curve  $H_{\mathbb{C}}$  for one equilibrium) are situated on  $H$ . In our case there are twenty two regions corresponding to topologically equivalent dynamical systems. The quoted curves determine the parametric portrait (Fig.1), therefore, this portrait consists of the union of  $S$ ,  $H$  and  $D$  for all equilibria.

Let  $\mathbf{x}_0$  be an equilibrium and denote by  $H_{\mathbb{R}}$ , the subset of  $H$  corresponding to real eigenvalues for  $\mathbf{x}_0$ , by  $H_{\mathbb{C}}$  the subset of  $H$  corresponding to purely imaginary eigenvalues for  $\mathbf{x}_0$ , and let  $Q$  be the intersection of  $H$  and  $S$ . Then, if  $\mathbf{x}_0$  exists for  $Q$ , the manifolds  $S$  and  $H$  are tangent at  $Q$ . Therefore  $H = H_{\mathbb{R}} \cup H_{\mathbb{C}}$  for every equilibria, even if  $H_{\mathbb{R}}$  and  $H_{\mathbb{C}}$  are not the same for any two equilibria. If  $\mathbf{x}_0$  does not exist for  $Q$ , then instead of  $Q$  we write  $C_3$  and remark that at  $C_3$  the curves  $H$  and  $S$  intersect, but not tangently.

The equations of the curves  $S$ ,  $H$ ,  $H_{\mathbb{C}}$ ,  $H_{\mathbb{R}}$  and  $D$  for the equilibrium point  $E$  are defined by the conditions  $\text{tr}\mathbf{A} \neq 0$ ,  $\det \mathbf{A} = 0$ ;  $\text{tr}\mathbf{A} = 0$ ;  $\text{tr}\mathbf{A} = 0$ ,  $\text{tr}^2 \mathbf{A} - 4 \det \mathbf{A} < 0$ ;  $\text{tr}\mathbf{A} = 0$ ,  $\text{tr}^2 \mathbf{A} - 4 \det \mathbf{A} > 0$ ; and  $\text{tr}^2 \mathbf{A} - 4 \det \mathbf{A} = 0$  respectively, where  $\mathbf{A}$  is given by (11). Thus, we obtain  $S = \{(\alpha, \varepsilon) \mid 1 - \alpha - \varepsilon = 0\}$ . From Section 2 we have that  $H = \{(\alpha, \varepsilon) \mid -\alpha^2 + \alpha - \alpha\varepsilon - \varepsilon = 0\}$ ,  $C_3 = H \cap S$ ,  $H_{\mathbb{R}} = \emptyset$  and  $H_{\mathbb{C}} = H \setminus C_3$ . Thus, in our case, for  $\alpha > 0$  and  $\varepsilon > 0$ , we have  $H_{\mathbb{C}} = \{(\alpha, \varepsilon) \mid -\alpha^2 + \alpha - \alpha\varepsilon - \varepsilon = 0\}$ . Of course, for the biologically nonrealistic case  $\varepsilon < 0$ , we have  $H_{\mathbb{R}} \neq \emptyset$  just as in the general case.

In Fig. 2 we represent the phase portraits corresponding to each stratum of the parametric portrait. They show that, in spite of their unrealistic significance for the population dynamics, the equilibria  $O$ ,  $E$ ,  $B$  and  $A$  for  $1 - \alpha - \varepsilon < 0$  heavily contribute to the changes in the phase portraits and, so, to the dynamic bifurcation diagram (which consists of Figs. 1 and 2).

It can be described as follows: the equilibrium  $O$  is always a saddle and it exists for every value of the parameters  $\alpha$  and  $\varepsilon$ ; the equilibrium  $B$  is always an attractive node and it exists for  $\alpha \neq 0$ ,  $\varepsilon \in \mathbb{R}$ , but it is not important for us, because it has a negative coordinate; the equilibrium  $E$  exists for  $\alpha \neq 0$  and  $\varepsilon \in \mathbb{R}$ . It is a saddle for  $1 - \alpha - \varepsilon \in (0, 1)$ , for  $1 - \alpha - \varepsilon = 1$  it collides with  $A$  and becomes a saddle-node, and then it is an attractive node for  $1 - \alpha - \varepsilon > 1$ . The equilibrium  $A$  exists for  $\alpha \neq 1$  and  $\varepsilon \in \mathbb{R}$ . For  $\alpha = \varepsilon = 0$  the equilibrium  $A$  corresponds to a linear center. On  $O\varepsilon$  it is an attractive focus for  $\varepsilon \in (0, 2\sqrt{2} - 2)$ , a sink for  $\varepsilon = 2\sqrt{2} - 2$ , an attractive node for  $\varepsilon \in (2\sqrt{2} - 2, 1)$ , for  $\varepsilon = 1$  it collides with  $E$  and becomes a saddle-node, and, then, for  $\varepsilon > 1$  it becomes a saddle. On  $O\alpha$  the equilibrium  $A$  is a repulsive focus for  $\alpha \in (0, 2\sqrt{2} - 2)$ , a source for  $\alpha = 2\sqrt{2} - 2$ , a repulsive node for  $\alpha \in (2\sqrt{2} - 2, 1)$ , it disappears for  $\alpha = 1$  and appears

again for  $\alpha > 1$  where it is a saddle, but it has no biological importance because one of its coordinates becomes negative.

For  $\alpha \neq 0, \varepsilon \neq 0$ , let us begin our discussion from the zone 9. Thus, in zone 9, the equilibrium  $A$  is a repulsive node; on the curve  $D$  (zone 10) it becomes a source and then, in zone 12 it is a repulsive focus. Somewhere in zone 12, it appears a limit cycle, which disappears on the curve  $H$  (zone 13) where  $A$  becomes a Hopf singularity, and then, in zone 14 it becomes an attractive focus. On the curve  $D$  (zone 15) the equilibrium  $A$  is a sink, then in zone 16 it becomes an attractive node and, then on the curve  $S$  (zone 17) it collides with the point  $E$  and becomes a nondegenerated saddle-node. Beyond the curve  $S$ , the equilibrium  $A$  becomes a saddle (but with the negative components), which disappears for  $\alpha = 1$  (zone 5) and appears again for  $\alpha > 1$  (zone 6). In Fig. 2 we present two portraits for zone 12 (one is global and one is local around the equilibrium  $A$ ).

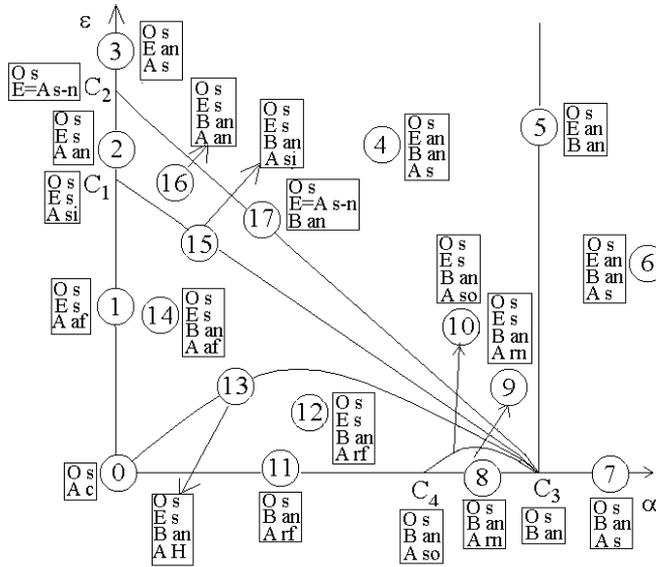


FIGURE 1. The parametric portrait for (6), where  $n$  stays for node,  $s$  for saddle,  $f$  for focus,  $c$  for center,  $s-n$  for saddle-node,  $si$  for sink,  $so$  for source,  $H$  for Hopf singularity,  $a$  for attractive and  $r$  for repulsive.

### 6. Biological interpretation

As we said before, for mathematical reasons we studied the population dynamics for the entire parameter plane. However, for biology purposes we give the biological interpretation only for the first quadrant. As a consequence, as we already established by convention, if  $x_0$  and/or  $y_0$  are negative we say that the corresponding equilibrium  $x_0$  does not exist.

Analyzing the portraits in Fig. 2 we can conclude that, if one or both initial populations do not exist, they will not exist for ever. In fact, this is a consequence of the fact that

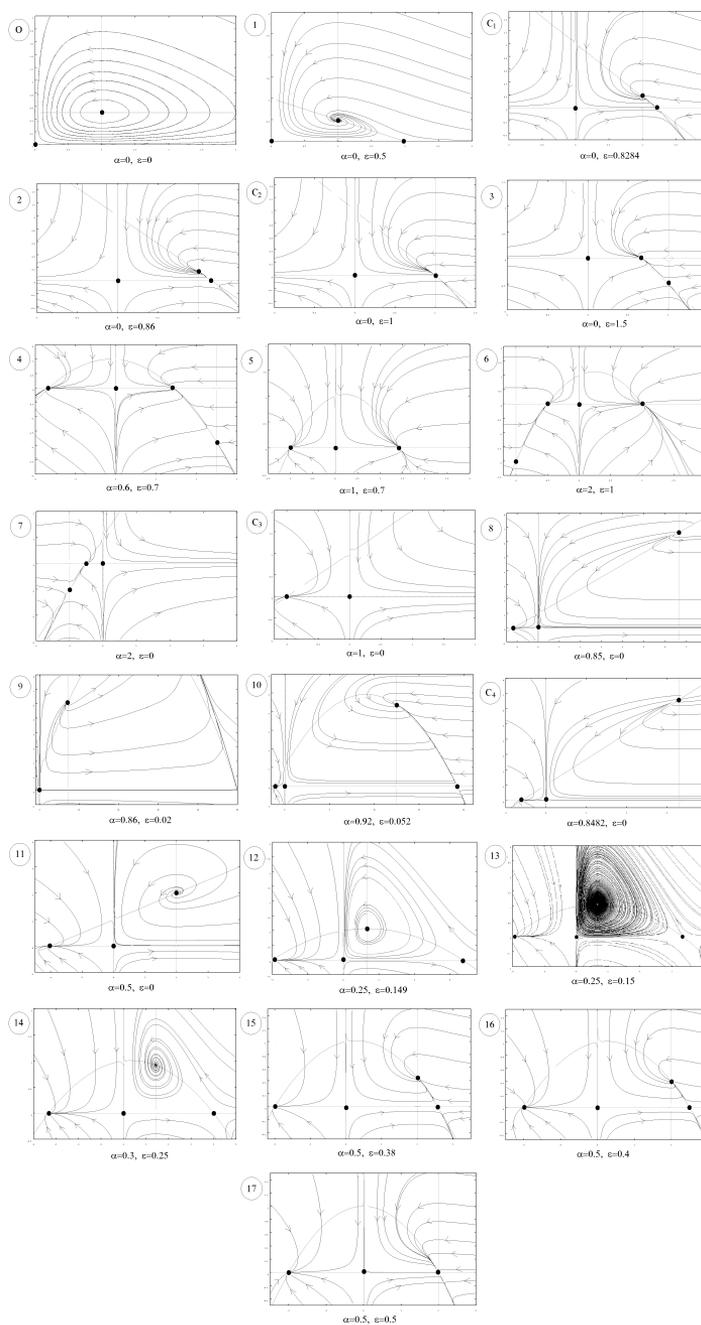


FIGURE 2. Phase portrait for various strata in Fig. 1.

the axes of coordinates in the phase plane are separatrix. If the initial population are at any equilibrium point, then the populations remain constant at any subsequent time. If the initial populations are situated on a limit cycle, then the time evolution of populations will be cyclical (periodic). For all other initial values the subsequent populations vary in various manners, depending on the values of the parameters  $\alpha$  and  $\varepsilon$ . The paths described by these initial values are phase space trajectories corresponding to transient regimes between some equilibrium state or/and periodic regime. In the following we are concerned only with the transient regimes.

Thus, for zone 0 the subsequent populations initially close to  $A$  vary periodically in time; for zones 1, 13 and 14 the subsequent populations are oscillatory but not periodic ( first both populations increase, then only  $y$  increases and  $x$  decreases, then only  $x$  increases and  $y$  decreases with the amplitudes smaller and smaller, and again both populations increase and so on) until they reach the equilibrium point  $A(1, 1 - \varepsilon)$  or  $A((1 - \alpha)^{-1}, (1 - \alpha - \varepsilon)(1 - \alpha)^{-2})$ . In zone 1 the increase (decrease) of the populations is faster than in zone 14, while in zone 13 the increase (decrease) of the populations is very slow. For zones  $C_1, 2, 15, 16$ , the subsequent populations present very small oscillations until they come to the equilibrium point  $A(1, 1 - \varepsilon)$  or  $A((1 - \alpha)^{-1}, (1 - \alpha - \varepsilon)(1 - \alpha)^{-2})$ . For zone 12 the subsequent populations oscillate (either periodically or not). For zones  $C_2, 3, 4, 5, 6, 17$ , the subsequent populations are slowly oscillating until they come to the equilibrium point  $E(1/\varepsilon, 0)$ , i.e.  $y$  becomes extinct. In all other cases the equilibrium points are unstable, namely one or both populations go to infinity.

Finally, we can conclude that the prey  $x(t)$  flourish in the absence of the predator. Theoretically, the predator can destroy all the prey so that the latter becomes extinct. However, if this happens the predator  $y(t)$  will also become extinct since, as we assume, it depends on the prey for its existence. In order for the predation to take place there must be a fight between a predator and a prey. In addition, our parametric portrait shows precisely where all these phenomena occur.

## References

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