

NONLINEAR WAVES IN AN ULTRARELATIVISTIC HEAT-CONDUCTING FLUID II (ECKART FORMULATION)

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ABSTRACT. In this paper a second-order theory for relativistic heat-conducting fluids is derived in the Eckart scheme, based on the assumption that the entropy 4-current should include quadratic terms in the heat flux. In the special case of ultrarelativistic fluids, the velocities of hydrodynamic and thermal weak discontinuity wave fronts are determined and, through the second-order compatibility conditions, the discontinuities associated to the waves and the transport equations for the amplitude of the discontinuities are found out. Finally, for heat wave, plane, cylindrical and spherical diverging waves are also investigated.

1. Introduction

Relativistic irreversible thermodynamics has a rather peculiar history. The first two proposals of relativistic dissipative fluid dynamics came from C. Eckart [1] in 1940 and from L. Landau and E. M. Lifshitz in the early fifties [2]. The difference in formal appearance is due to the different choices for the definition of the hydrodynamical 4-velocity. One of the popular discussions of cosmology is given by fluid approximation [3, 4]. This phenomenological approach to cosmological models requires the adoption of a macroscopic 4-velocity representative of the continuum flow. As we say, we have two natural specifications for this hydrodynamic field reflecting: the average flux of the constituent particles of the fluid (the Eckart or particle frame) [1], or the average flux of total energy (the Landau-Lifshitz or energy frame) [2]. These frames are formally defined as being the normalized vector parallel to the particle flux and the normalized timelike eigenvector of the energy-momentum tensor, respectively. The two choices have different computational advantages. The Landau-Lifshitz formalism is convenient since it reduces the energy-momentum tensor to a simpler form. Instead, the Eckart formalism has the advantage of a simpler integration of particle conservation law.

This conventional theories of dissipative fluid dynamics are based on the assumption that the entropy 4-current contains terms up to first order in dissipative quantities and hence they are referred to as *first order theories* of dissipative fluids [5]. The resulting equations for the dissipative fluxes are linear in the thermodynamic forces, and the resulting equations of motion are parabolic in structure. It was later shown by W. Israel and coworkers [6, 7]

that both theories have the undesirable feature that causality principle may not be satisfied. That is, thermal signals may propagate with speed exceeding that of light.

In order to solve this feature, extended theories of dissipative fluid were introduced.

These causal theories are based on the assumption that the entropy 4-current should include quadratic terms in the dissipative fluxes and hence they are referred to as *second order theories*.

In particular two paths can be followed: the extended irreversible thermodynamics developed by Jou [8], Müller [9] and Ruggeri [10], but also Galipò [11] and Giambò [12], allowing the inclusion of the dissipative quantities in the expression of the entropy density and the entropy flux due to a generalized Gibbs relation; or, following Muronga [13], Israel and Stewart [6, 7, 14], a model that introduces additional dynamical fields through the assumption that the entropy 4-current includes quadratic terms in the heat flux. This model has been taken up and developed later by Garcia-Colin and Sandoval-Villalbalzo [15], Garcia-Perciante et al. [16], Samuelsson et al. [17]. There is also an important approach to heat conduction, in which one considers a multi-fluid system whose species are represented by a particle number density current and an entropy flux, in general not aligned with the particle flux [18]. This model has been recovered and developed recently by Lopez-Monsalvo and Andersson [19] and Andersson and Comer [20]. The resulting equations for the dissipative fluxes are hyperbolic and they lead to causal propagation of signals [6, 7, 13, 14, 21, 22, 23].

In second order theories the space of thermodynamic quantities is extended to include the dissipative quantities which are treated as field variables as well.

This last approach is the base of the model presented in this paper. Our aim is to deduce a second order theory for relativistic heat-conducting fluids, in the Eckart scheme, based on the assumption that the entropy 4-current includes quadratic terms in heat flux and to determine, in the special case of ultrarelativistic fluids, the velocities of hydrodynamical and thermal weak discontinuity waves.

More precisely, this paper is outlined as follows. In Section 2 the equilibrium thermodynamics of relativistic perfect fluids is described. In Section 3 the formulation of relativistic heat-conducting fluid model in Eckart scheme is introduced. A second order theory following the second approach in the Landau-Lifshitz scheme can be found in [24]. In Section 4 the propagation of weak discontinuity is investigated. In Section 5 the special case of ultrarelativistic fluids is considered. In Section 6 the transport equation describing the evolution, along the rays, of the amplitude of the discontinuities is determined. In particular, the hydrodynamic wave is discussed and the thermal wave, in special case of plane, cylindrical and spherical diverging waves is investigated.

Throughout this paper, a coordinate system x^α , being $x^0 = t$ the time and x^i the spatial coordinates in the flat space-time of special relativity is introduced. The fundamental metric tensor $g_{\alpha\beta}$ is defined by: $g_{00} = 1$, $g_{ii} = -1$, $g_{\alpha\beta} = 0$, for $\alpha \neq \beta$, where the velocity of light in the vacuum, c , is set equal to 1.

Greek indices range over 0, 1, 2, 3 and the Latin ones over 1, 2, 3.

The notation $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ represents the partial derivative with respect to x^α .

2. Basic of equilibrium fluid dynamics

When we study the equilibrium thermodynamics three important variables must be taken into account: the energy density ρ , the particle number density r and the specific entropy S . The energy density and the particle number density are related by

$$\rho = r(1 + \varepsilon), \quad (1)$$

where ε is the specific internal energy.

These basic quantities will be referred to as primary thermodynamic variables, from which to deduce all other state variables, such as the pressure p .

From the equation of state for the entropy density $s = rS$, $s = s(\rho, r)$, and Euler relation

$$\mu = \frac{\rho + p}{r} - TS, \quad (2)$$

defining chemical potential μ , the fundamental Gibbs equation can be written as:

$$Tds = d\rho - \mu dr, \quad (3)$$

where T denotes the temperature.

Eq. (2) defines the last unknown thermodynamic function p .

In relativistic fluid dynamics it is necessary to operate with covariant object, so all the thermodynamic quantities are expressed in terms of the 4-vector number current R_{eq}^α , the energy-momentum tensor $T_{eq}^{\alpha\beta}$ and the entropy 4-current S_{eq}^α at equilibrium

$$R_{eq}^\alpha = ru^\alpha, \quad (4)$$

$$T_{eq}^{\alpha\beta} = \rho u^\alpha u^\beta - p\gamma^{\alpha\beta}, \quad (5)$$

$$S_{eq}^\alpha = rSu^\alpha, \quad (6)$$

where u^α is the unitary hydrodynamical 4-velocity ($u^\alpha u_\alpha = 1$), whereas $\gamma^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ is the spatial projection tensor orthogonal to u^α .

Therefore, the equilibrium state is described by five independent variables ρ, r, u^α .

The thermodynamic relation (3) can be rewritten in terms of the covariant quantities R_{eq}^α , $T_{eq}^{\alpha\beta}$ and S_{eq}^α

$$dS_{eq}^\alpha = -\frac{\mu}{T}dR_{eq}^\alpha + \frac{1}{T}u_\beta dT_{eq}^{\alpha\beta}. \quad (7)$$

From eqs. (2)-(6) it follows immediately that

$$S_{eq}^\alpha = \frac{p}{T}u^\alpha - \frac{\mu}{T}R_{eq}^\alpha + \frac{1}{T}u_\beta T_{eq}^{\alpha\beta}, \quad (8)$$

$$d\left(\frac{p}{T}u^\alpha\right) = R_{eq}^\alpha d\left(\frac{\mu}{T}\right) - T_{eq}^{\alpha\beta} d\left(\frac{u_\beta}{T}\right). \quad (9)$$

So the covariant form of particle, energy, momentum and entropy conservation equations reads as

$$\partial_\alpha R_{eq}^\alpha = 0, \quad \text{i.e.} \quad u^\alpha \partial_\alpha r + r \partial_\alpha u^\alpha = 0, \quad (10)$$

$$\partial_\beta T_{eq}^{\alpha\beta} = 0, \quad \text{i.e.} \quad (\rho + p)u^\beta \partial_\beta u^\alpha + u^\alpha u^\beta \partial_\beta \rho - \gamma^{\alpha\beta} \partial_\beta p = 0, \quad (11)$$

$$\partial_\alpha S_{eq}^\alpha = 0, \quad \text{i.e.} \quad u^\alpha \partial_\alpha (rS) + rS \partial_\alpha u^\alpha = 0, \quad (12)$$

where the spatial projection of (11) and its projection along u^α are, respectively,

$$\gamma_\beta^\alpha \partial_\lambda T_{eq}^{\beta\lambda} = 0, \quad \text{i.e.} \quad (\rho + p) u^\beta \partial_\beta u^\alpha - \gamma^{\alpha\beta} \partial_\beta p = 0, \quad (13)$$

$$u_\alpha \partial_\beta T_{eq}^{\alpha\beta} = 0, \quad \text{i.e.} \quad u^\alpha \partial_\alpha \rho + (\rho + p) \partial_\alpha u^\alpha = 0. \quad (14)$$

3. Non-equilibrium states

Non-equilibrium effects are introduced by enlarging the space of independent variables through the introduction of non-equilibrium variables.

In principle, the inclusion of dissipative processes as heat conduction requires additional terms in the primary variables R_{eq}^α , $T_{eq}^{\alpha\beta}$ and S_{eq}^α describing a perfect fluid.

The next step is to find evolution equations for these extra variables. Whereas the evolution equations for the equilibrium variables are given by the usual conservation laws, general criteria do not exist concerning the evolution equations of the dissipative fluxes, with the exception of the restriction imposed on them by the second law of thermodynamics.

However, the presence of a heat transfer involves a problem regarding the definition of the hydrodynamical 4-velocity u^α . In Eckart's formulation, u^α is identified by the 4-velocity of particle transport (particle frame) [1]. Formally, the particle frame is the unique time-like vector parallel to R_{eq}^α .

In presence of irreversible processes as the heat conduction, (4) preserves the same structure, while small terms $\Delta T^{\alpha\beta}$ and ΔS^α have to be added in (5) and (6),

$$T^{\alpha\beta} = T_{eq}^{\alpha\beta} + \Delta T^{\alpha\beta}, \quad (15)$$

$$S^\alpha = S_{eq}^\alpha + \Delta S^\alpha, \quad (16)$$

such that the corresponding conservation laws are satisfied

$$\partial_\alpha R^\alpha = 0, \quad (17)$$

$$\partial_\alpha T^{\alpha\beta} = 0, \quad (18)$$

and the second law of thermodynamics holds

$$\partial_\alpha S^\alpha \geq 0. \quad (19)$$

A non-equilibrium state is characterized by increasing entropy due to the presence of dissipative fluxes.

The deviations $\Delta T^{\alpha\beta}$ and ΔS^α from local equilibrium contain the information about heat flux and 4-current entropy at non-equilibrium state.

From eq. (18) it follows that

$$\partial_\alpha T^{\alpha\beta} = \partial_\alpha T_{eq}^{\alpha\beta} + \partial_\alpha (\Delta T^{\alpha\beta}) = 0. \quad (20)$$

Now, by virtue of eqs. (14) and (10) and taking into account the equation for covariant derivatives along the world lines following from Gibbs relation (3), the following equation holds the covariant derivative of eq. (3) along the world lines of the fluid defined by u^α , we obtain

$$T \partial_\alpha (r S u^\alpha) = -u_\beta \partial_\alpha (\Delta T^{\alpha\beta}). \quad (21)$$

The non-equilibrium entropy 4-current $S^\alpha = S^\alpha(R^\alpha, T^{\alpha\beta})$ takes the form [14]:

$$S^\alpha = \frac{p}{T}u^\alpha - \frac{\mu}{T}R^\alpha + \frac{1}{T}u_\beta T^{\alpha\beta} + Q^\alpha (\Delta T^{\alpha\beta}) , \quad (22)$$

where Q^α is a function of deviation $\Delta T^{\alpha\beta}$.

For small deviations, in order to obtain equations guaranteeing causality and hyperbolicity, it is sufficient to keep only quadratic terms in the Taylor's expansion of Q^α .

By virtue of eqs. (2), (4) and (15), together with the relations [25]:

$$\Delta T^{\alpha\beta} = q^\alpha u^\beta + q^\beta u^\alpha , \quad q^\alpha u_\alpha = 0 , \quad q_\alpha q^\alpha < 0 , \quad (23)$$

the most general algebraic form for S^α , at most of second order in the dissipative fluxes $\Delta T^{\alpha\beta}$, is [6, 7, 14, 25]:

$$S^\alpha = rS u^\alpha + \frac{1}{T}q^\alpha + k_E (q^\mu q_\mu) u^\alpha , \quad (24)$$

where k_E is a thermodynamic coefficient (which in general is a function of the state variables) accounting for dissipative contribution to the entropy density and q^α is the heat flux.

In what follows, in order to simplify the notation, the thermodynamic coefficient will be denoted only k .

From eq. (24) it follows that the effective entropy density measured by comoving observer is

$$u_\alpha S^\alpha = rS - kq^2 , \quad (25)$$

where $q^2 = -q^\alpha q_\alpha$. Since the entropy density is maximum at equilibrium, the condition $u_\alpha Q^\alpha \leq 0$ (or, equivalently, $q^\alpha q_\alpha < 0$) implies that k is nonnegative.

The divergence of extended current (24), together with eq. (21) in which eq. (23)₁ have been taken into account, leads to

$$\partial_\alpha S^\alpha = -\frac{1}{T^2}q^\alpha [\partial_\alpha T - 2kT^2 u^\lambda \partial_\lambda q_\alpha - T^2 q_\alpha \partial_\lambda (ku^\lambda) - Tu^\lambda \partial_\lambda u_\alpha] \geq 0 . \quad (26)$$

Since $q^\alpha q_\alpha < 0$, from eq. (26) it follows that the heat flux is given by

$$q_\alpha = -\chi \gamma_\alpha^\beta \{ \partial_\beta T - 2kT^2 u^\lambda \partial_\lambda q_\beta - T^2 q_\beta \partial_\lambda (ku^\lambda) - Tu^\lambda \partial_\lambda u_\beta \} , \quad (27)$$

where the phenomenological coefficient $\chi (\geq 0)$ is the thermal conductivity of the fluid.

We can conclude that, in the Eckart scheme, the set of hyperbolic equations describing the motion of a relativistic heat-conducting fluid is:

$$\begin{aligned} r\partial_\alpha u^\alpha + u^\alpha \partial_\alpha r &= 0 , \\ (\rho + p)u^\alpha \partial_\alpha u^\beta - \gamma^{\alpha\beta} \partial_\alpha p + u^\alpha \partial_\alpha q^\beta + q^\alpha \partial_\alpha u^\beta \\ &\quad + q^\beta \partial_\alpha u^\alpha + u^\alpha u^\beta q^\lambda \partial_\alpha u_\lambda = 0 , \\ rTu^\alpha \partial_\alpha S + \partial_\alpha q^\alpha - u^\alpha q^\beta \partial_\alpha u_\beta &= 0 , \\ q_\alpha &= -\chi \gamma_\alpha^\beta \{ \partial_\beta T - 2kT^2 u^\lambda \partial_\lambda q_\beta - T^2 q_\beta \partial_\lambda (ku^\lambda) - Tu^\lambda \partial_\lambda u_\beta \} , \end{aligned} \quad (28)$$

in the 8 independent field variables r, T, u^α, q^α .

If we assume r and T as state variables, then $p = p(r, T)$ and $S = S(r, T)$. This means that

$$\partial_\alpha p = \left(\frac{\partial p}{\partial r} \right)_T \partial_\alpha r + \left(\frac{\partial p}{\partial T} \right)_r \partial_\alpha T , \quad (29)$$

and

$$\partial_\alpha S = \left(\frac{\partial S}{\partial r}\right)_T \partial_\alpha r + \left(\frac{\partial S}{\partial T}\right)_r \partial_\alpha T. \tag{30}$$

4. Weak discontinuities propagation

Consider now a propagating time-like, singular hypersurface Σ defined by

$$\varphi(x^\alpha) = 0, \quad x^\alpha = x^\alpha(w^A), \tag{31}$$

where w^A , $A = 0, 1, 2$, are the coordinates on Σ .

Let a_{AB} and b_{AB} be, respectively, the components of the first and second fundamental covariant tensors of Σ . Let N^α be the space-like unit normal vector to Σ .

Now let us recall the following relations [26]:

$$\begin{aligned} a_{AB} &= g_{\alpha\beta} x^\alpha_{,A} x^\beta_{,B}, & N_\alpha N^\alpha &= -1, \\ N_\alpha x^\alpha_{,A} &= 0, & x^\alpha_{,AB} &= b_{AB} N^\alpha, \\ N_{,\alpha} &= a^{AB} b_{AB} = b^A_A, & N^\alpha_{,A} &= b^B_A x^\alpha_{,B}, \\ a^{AB} x^\alpha_{,A} x^\beta_{,B} &= g^{\alpha\beta} + N^\alpha N^\beta, & x_{\alpha,A} &= g_{\alpha\beta} x^\beta_{,A}. \end{aligned} \tag{32}$$

In the above relations, a comma followed by a Greek index denotes the spatial derivative with respect to x^α , whereas a comma followed by a Latin index denotes covariant derivative with respect to a_{AB} ; since x^α are scalar functions of w^A , we have

$$x^\alpha_{,A} = \frac{\partial x^\alpha}{\partial w^A}. \tag{33}$$

Making use of Hadamard’s method for characteristic hypersurfaces of possible discontinuity, we investigate the problem of propagation of discontinuity surfaces of first order compatible with system (28), in which the algebraically related variables r , T , u^α , q^α are continuous, but their space and time derivatives exhibit jump discontinuities across a characteristic hypersurface Σ , with tangent direction specified by the normal co-vector of components N^α . This means that the surface Σ can be interpreted as the wave front of a propagating weak discontinuity.

The state ahead of the wave Σ is assumed to be uniform and in complete equilibrium.

Let us recall the compatibility conditions, derived by Hadamard’s Lemma, which must be satisfied across Σ by the partial derivatives of the field variables [27, 28, 29, 30]. For the first and second partial derivatives, under the assumption that Σ is a weak discontinuity surface (the first order derivatives of field variables are discontinuous across Σ), they read as

$$[F_{,\alpha}] = \nu N_\alpha, \tag{34}$$

$$[F_{,\alpha\beta}] = \bar{\nu} N_\alpha N_\beta + a^{AB} \nu_{,A} (N_\alpha x_{\beta,B} + N_\beta x_{\alpha,B}) + \nu b^{AB} x_{\alpha,A} x_{\beta,B}, \tag{35}$$

where F is any field variable, the square bracket denotes the jump across Σ , *i.e.*, $[F] = F_2 - F_1$, subscript 1 and 2 denoting the limiting values of F from each side of the surface

Σ and ν and $\bar{\nu}$ denote the discontinuities in the normal derivatives. Also $\frac{dF}{d\sigma}$ represents the rate of change of F as seen by an observer comoving with Σ .

At this point, the following quantities defined on Σ can be introduced:

$$\begin{aligned} \left[\frac{\partial r}{\partial x^\alpha} \right] N^\alpha &= -\xi, & \left[\frac{\partial T}{\partial x^\alpha} \right] N^\alpha &= -\vartheta, \\ \left[\frac{\partial u^\beta}{\partial x^\alpha} \right] N^\alpha &= -\omega^\beta, & \left[\frac{\partial q^\beta}{\partial x^\alpha} \right] N^\alpha &= -\pi^\beta, \end{aligned} \tag{36}$$

representing the jumps in the normal derivatives of r , T , u^α and q^α , respectively.

With respect to a rest frame determined by some preferred time-like unit vector u^α , the velocity, λ , of propagation in the direction of an orthogonal unit space-like vector n^α will be given, for a suitably normalized N^α , by

$$\lambda = \frac{L}{\ell}, \quad n^\alpha = \frac{1}{\ell} (N^\alpha - L u^\alpha), \tag{37}$$

where

$$L = u^\alpha N_\alpha, \quad \ell^2 = 1 + L^2. \tag{38}$$

In what follows the medium ahead of Σ is supposed to be uniform and at rest.

In this way, if, as usual [5, 31, 32], small perturbations of the thermal equilibrium are considered in which no relative transport occurs, then it can be chosen

$$q^\alpha = 0 \tag{39}$$

along the unperturbed flow direction.

Now, the first-order compatibility conditions (34) and condition (39) can be applied to system (28). Noticing that from (23)₂, with assumption (39), it follows that

$$u_\alpha \pi^\alpha = u_\alpha \omega^\alpha = 0,$$

we obtain:

$$\left\{ \begin{aligned} L\xi + r\omega_N &= 0, \\ rfL\omega^\beta - \ell n^\beta \left[\left(\frac{\partial p}{\partial r} \right)_T \xi + \left(\frac{\partial p}{\partial T} \right)_r \vartheta \right] + L\pi^\beta &= 0, \\ rTL \left(\frac{\partial S}{\partial r} \right)_T \xi + rTL \left(\frac{\partial S}{\partial T} \right)_r \vartheta + \pi_N &= 0, \\ \gamma_\alpha^\beta \vartheta N_\beta - TL\omega_\alpha - 2kT^2 L\pi_\alpha &= 0. \end{aligned} \right. \tag{40}$$

Multiplying (40)₂ by N_β and (40)₄ by N^α the following equations, respectively, are obtained

$$\ell^2 \left(\frac{\partial p}{\partial r} \right)_T \xi + \ell^2 \left(\frac{\partial p}{\partial T} \right)_r \vartheta + rfL\omega_N + L\pi_N = 0 \tag{41}$$

and

$$\ell^2 \vartheta + TL\omega_N + 2kT^2 L\pi_N = 0, \tag{42}$$

where ω_N and π_N are the normal components of the two vectors, ω and π , on Σ .

System (40)₁, (40)₃, (41), (42) admits nontrivial solutions in ξ , ϑ , ω_N and π_N if, and only if, the determinant of the coefficient matrix vanishes.

In the special case in which $p = p(\rho) = p(\rho(r, T))$, by virtue of eqs. (2) and (3), the following relations hold

$$\begin{aligned} \left(\frac{\partial p}{\partial r}\right)_T &= \left(\frac{dp}{d\rho}\right) \left(\frac{\partial \rho}{\partial r}\right)_T = p' \left[f + rT \left(\frac{\partial S}{\partial r}\right)_T \right], \\ \left(\frac{\partial p}{\partial T}\right)_r &= \left(\frac{dp}{d\rho}\right) \left(\frac{\partial \rho}{\partial T}\right)_r = p' rT \left(\frac{\partial S}{\partial T}\right)_r, \end{aligned} \tag{43}$$

where $f = (\rho + p)/r$ is the fluid index.

The following *characteristic equation* for the velocity of propagation $\lambda = L/\ell$ is obtained:

$$(L^2 - p'\ell^2) \left[rT^2 \left(\frac{\partial S}{\partial T}\right)_r (1 - 2krTf) L^2 + r \left(f + rT \left(\frac{\partial S}{\partial r}\right)_T \right) \ell^2 \right] = 0. \tag{44}$$

This equation implies the existence of the two expected well-behaved propagation modes, which are interpretable as

- a *hydrodynamic wave* with velocity given by $\lambda_1^2 = p'$,
- a *heat wave* with velocity given by $\lambda_2^2 = \frac{f + rTS'_r}{T^2 S'_T (2krTf - 1)}$,

where

$$S'_r = \left(\frac{\partial S}{\partial r}\right)_T, \quad S'_T = \left(\frac{\partial S}{\partial T}\right)_r.$$

In addition, from system (40), the solution $L = 0$, which represents a wave moving with the fluid, is obtained. The corresponding discontinuities satisfy the equations

$$\begin{cases} \xi = 0, \\ \vartheta = 0, \\ \omega_N = 0, \\ \pi_N = 0. \end{cases} \tag{45}$$

Since the coefficients characterizing the discontinuities exhibit four degrees of freedom, then system (40) admits four independent eigenvectors corresponding to $L = 0$ in the space of the field variables.

5. Ultrarelativistic fluids

In the case of ultrarelativistic fluid the energy density and the particle number density are related only by

$$\rho = r\varepsilon, \tag{46}$$

because $r \ll r\varepsilon$. This relation implies that the pressure law is:

$$p = \rho(\gamma - 1), \tag{47}$$

where $\gamma = C_p/C_V$ is the ratio of specific heats.

In particular, in the case of perfect politropic fluid, the state equation reads as

$$p = rRT, \tag{48}$$

in which $R = C_p - C_V$ is the universal gas constant and then

$$\begin{aligned}
 p' = \frac{dp}{d\rho} &= \frac{R}{C_V}, & \left(\frac{\partial p}{\partial r}\right)_T &= RT, & \left(\frac{\partial p}{\partial T}\right)_r &= rR, \\
 \left(\frac{\partial S}{\partial r}\right)_T &= -\frac{R}{r}, & \left(\frac{\partial S}{\partial T}\right)_r &= \frac{C_V}{T}, & f &= (R + C_V)T = C_p T.
 \end{aligned} \tag{49}$$

By virtue of relations (49), system (40)₁, (40)₃, (41), (42) can be written as

$$\begin{cases}
 L\xi + r\omega_N = 0, \\
 RT\ell^2\xi + rR\ell^2\vartheta + rfL\omega_N + L\pi_N = 0, \\
 -RTL\xi + rC_V L\vartheta + \pi_N = 0, \\
 \ell^2\vartheta + TL\omega_N + 2kT^2L\pi_N = 0,
 \end{cases} \tag{50}$$

and the velocities of propagation write as

$$\lambda_1^2 = \frac{R}{C_V}, \quad \lambda_2^2 = \frac{1}{2krTf - 1}. \tag{51}$$

6. Discontinuity transport equation

Now we determine the discontinuities associated to the first and second sound and the *transport equation* describing the evolution, along the rays, of the amplitude, ψ , of the discontinuities, for an ultrarelativistic fluid.

In order to deduce the transport equation, the discontinuities in the second order partial derivatives along the normal vector of particle number density r , temperature T , unitary hydrodynamical 4-velocity u^α and heat flux q^α must be introduced and are denoted by

$$\begin{aligned}
 \left[\frac{\partial^2 r}{\partial x^\alpha \partial x^\beta}\right] N^\alpha N^\beta &= \bar{\xi}, & \left[\frac{\partial^2 T}{\partial x^\alpha \partial x^\beta}\right] N^\alpha N^\beta &= \bar{\vartheta}, \\
 \left[\frac{\partial^2 u^\mu}{\partial x^\alpha \partial x^\beta}\right] N^\alpha N^\beta &= \bar{\omega}^\mu, & \left[\frac{\partial^2 q^\mu}{\partial x^\alpha \partial x^\beta}\right] N^\alpha N^\beta &= \bar{\pi}^\mu.
 \end{aligned} \tag{52}$$

Furthermore, we assume that $k = k(r, T)$.

Now, differentiating system (28) with respect to x^γ , computing the jumps, using (52) and the second-order compatibility conditions (35) and then multiplying by N^γ , following system is obtained:

$$\begin{cases}
 L\bar{\xi} + r\bar{\omega}_N = A, \\
 RT\ell^2\bar{\xi} + rR\ell^2\bar{\vartheta} + rfL\bar{\omega}_N + L\bar{\pi}_N = B, \\
 -RTL\bar{\xi} + rC_V L\bar{\vartheta} + \bar{\pi}_N = C, \\
 \ell^2\bar{\vartheta} + TL\bar{\omega}_N + 2kT^2L\bar{\pi}_N = D,
 \end{cases} \tag{53}$$

where

$$\begin{aligned}
 A &= -\ell \frac{d\xi}{d\sigma} - r a^{AB} \omega_{,A}^\alpha x_{\alpha,B} - 2\xi \omega_N, \\
 B &= - \left\{ r f \ell \frac{d\omega_N}{d\sigma} + RT L \ell \frac{d\xi}{d\sigma} + r R L \ell \frac{d\vartheta}{d\sigma} + \ell \frac{d\pi_N}{d\sigma} + 2R \ell^2 \xi \vartheta + L^2 \pi^\lambda \omega_\lambda \right. \\
 &\quad \left. + 3\pi_N \omega_N + r f \omega_N^2 + (f + 2RT) L \xi \omega_N + (r C_V + 3rR) L \vartheta \omega_N \right\}, \\
 C &= RT \ell \frac{d\xi}{d\sigma} - r C_V \ell \frac{d\vartheta}{d\sigma} - a^{AB} \pi_{,A}^\alpha x_{\alpha,B} + (R - C_V) L \xi \vartheta \\
 &\quad + RT \xi \omega_N - r C_V \vartheta \omega_N + L \pi^\beta \omega_\beta, \\
 D &= - \left\{ L \ell \frac{d\vartheta}{d\sigma} + 2k T^2 \ell \frac{d\pi_N}{d\sigma} + T \ell \frac{d\omega_N}{d\sigma} + 3k_r T^2 L \xi \pi_N + 3k_T T^2 L \vartheta \pi_N \right. \\
 &\quad + 4k T L \vartheta \pi_N + 3k T^2 \omega_N \pi_N + 3L \vartheta \omega_N \\
 &\quad \left. + T \omega_N^2 - 2k T^2 L^2 \omega^\beta \pi_\beta - T L^2 \omega_\beta \omega^\beta + \frac{1}{\chi} \pi_N \right\}.
 \end{aligned} \tag{54}$$

In equations (53)-(54), the derivative along the ray is used and for a given function F it is given by

$$\frac{dF}{d\sigma} = \frac{1}{\ell} a^{AB} u^\alpha x_{\alpha,B} F_{,A}, \tag{55}$$

where σ is the ray parameter and the ray derivatives of the jumps of unitary hydrodynamical 4-velocity u^α and heat flux q^α are given by

$$\frac{d\omega_N}{d\sigma} = \frac{d\omega_\beta}{d\sigma} N^\beta, \quad \frac{d\pi_N}{d\sigma} = \frac{d\pi_\beta}{d\sigma} N^\beta. \tag{56}$$

At this point, we are able to determine the transport equations for the two amplitudes ψ_1 and ψ_2 in the two wave modes.

Let us consider separately the two waves.

6.1. Hydrodynamic wave. For the first wave, from system (50), in which the velocity of propagation is $\lambda_1^2 = R/C_V$, the following system relative to the discontinuities associated to the wave is obtained:

$$\left\{ \begin{aligned}
 \xi &= \frac{r\ell}{L} \psi_1, \\
 \vartheta &= \frac{TL}{\ell} \psi_1, \\
 \pi_N &= 0, \\
 \pi_\alpha &= 0, \\
 \omega_\alpha &= \psi_1 n_\alpha,
 \end{aligned} \right. \tag{57}$$

where $\psi_1 = -\frac{1}{\ell} \omega_N$.

The transport equation for ψ_1 can now be deduced.

Using (32), (37)₂, (55), (56) and (57), we obtain the following relations:

$$a^{AB}\omega_{,A}^\alpha x_{\alpha,B} = -L\frac{d\psi_1}{d\sigma} + \frac{2}{\ell}\Omega\psi_1, \tag{58}$$

where Ω is the mean curvature of the hypersurface Σ and

$$\frac{d\xi}{d\sigma} = \frac{r\ell}{L}\frac{d\psi_1}{d\sigma}, \quad \frac{d\vartheta}{d\sigma} = \frac{TL}{\ell}\frac{d\psi_1}{d\sigma}, \quad \frac{d\omega_N}{d\sigma} = -\ell\frac{d\psi_1}{d\sigma}, \tag{59}$$

$$\xi\omega_N = -\frac{r\ell^2}{L}\psi_1^2, \quad \xi\vartheta = rT\psi_1^2, \quad \omega_N^2 = \ell^2\psi_1^2, \quad \vartheta\omega_N = -TL\psi_1^2. \tag{60}$$

Using (58)-(60) into system (53), the following transport equation for the amplitude ψ_1 is get:

$$\frac{d\psi_1}{d\sigma} + \frac{L}{\ell}\Omega\psi_1 - \psi_1^2 = 0. \tag{61}$$

In order to integrate equations (61), we introduce the “proper time”, τ , defined by

$$\ell d\sigma = d\tau \tag{62}$$

and equation (61) writes as

$$\frac{d\psi_1}{d\tau} + \frac{L}{\ell^2}\Omega\psi_1 - \frac{1}{\ell}\psi_1^2 = 0, \tag{63}$$

where

$$\frac{1}{\ell^2} = 2 - \gamma \Rightarrow \frac{1}{\ell} = \sqrt{2 - \gamma}.$$

If we put $a_0 = -\frac{L}{\ell^2} = -(2 - \gamma)L$, *i.e.* the constant speed of propagation of the wave multiplied by $\frac{1}{\ell}$, and $P_0 = -\sqrt{2 - \gamma}$, the transport equation for ψ_1 can be rewritten as

$$\frac{d\psi_1}{d\tau} - a_0\Omega\psi_1 + P_0\psi_1^2 = 0. \tag{64}$$

It has, therefore, that the amplitude of the discontinuity is governed by Bernoulli equation [33, 34].

In the present context, the quantities with subscript 0 appearing in (64) are evaluated in the local rest frame, so they are constants.

Since the mean curvature Ω at any point of the wave surface Σ admits the representation [35]

$$\Omega = \frac{\Omega_0 - k_0 a_0 \tau}{1 - 2\Omega_0 a_0 \tau + k_0 a_0^2 \tau^2}, \tag{65}$$

where Ω_0 and k_0 are the mean and Gaussian curvatures of Σ at $\tau = 0$, respectively, eq. (64) can be integrated to yield [36, 37]

$$\psi_1 = \frac{\psi_{01}(1 - 2a_0\Omega_0\tau + k_0a_0^2\tau^2)^{-1/2}}{1 + P_0\psi_{01}\int_0^\tau(1 - 2a_0\Omega_0\hat{\tau} + k_0a_0^2\hat{\tau}^2)^{-1/2}d\hat{\tau}}, \tag{66}$$

where ψ_{01} is the value of ψ_1 on the wave front at $\tau = 0$.

As can be easily noticed, the transport equation (64) is nonlinear. So a critical time may exist at which the weak discontinuity wave Σ evolves into a shock wave, in the sense that the amplitude ψ_1 of the discontinuity of the first-order derivatives becomes infinite as τ tends to the critical time, $\tau_c > 0$.

Consider now equation (64). In order to focus on the physical aspects, we discuss now the case of plane waves.

For a plane wave front $\Omega_0 = k_0 = 0$ and eq. (66) yields

$$\psi_1 = \frac{\psi_{01}}{1 + P_0\psi_{01}\tau}. \tag{67}$$

Eq. (67) shows that if $\psi_{01} > 0$ (i.e., an expansive wave front), a critical time appears

$$\tau_c = \frac{1}{\sqrt{2 - \gamma\psi_{01}}}, \tag{68}$$

at which the amplitude blows up (i.e., $\psi_1 \rightarrow \infty$ as $\tau \rightarrow \tau_c$). Thus, at the time τ_c the velocity gradient on the wave front becomes infinite and the weak discontinuity evolves into a shock wave.

As can be easily observed, for $\gamma = 2$ the solution $\psi_1 = \psi_{01}$ exists $\forall \tau \geq 0$.

Conversely, if $\psi_{01} < 0$ (i.e., a compressive wave front), the denominator of the eq. (67) does not vanishes for any τ and discontinuity decays since $\psi_1 \rightarrow 0$ as $\tau \rightarrow \infty$.

6.2. Heat wave. For the second wave the velocity of propagation is $\lambda_2^2 = 1/(2krTf - 1)$ and we obtain the following system for the discontinuities:

$$\left\{ \begin{array}{l} \xi = -\frac{\ell}{fL}\psi_2, \\ \vartheta = \frac{T\ell}{rfL}\psi_2, \\ \omega_N = \frac{\ell}{rf}\psi_2, \\ \omega_\alpha = -\frac{1}{rf}\psi_2 n_\alpha, \\ \pi_\alpha = \psi_2 n_\alpha, \end{array} \right. \tag{69}$$

where $\psi_2 = -\frac{1}{\ell}\pi_N$.

As for the first wave, we can deduce the transport equation for the amplitude ψ_2 .

Using (32), (37)₂, (55), (56) and (69), we obtain the following relations:

$$a^{AB}\pi_{,A}x_{\alpha,B} = -L\frac{d\psi_2}{d\sigma} + \frac{2}{\ell}\Omega\psi_2, \tag{70}$$

and

$$\frac{d\xi}{d\sigma} = -\frac{\ell}{fL}\frac{d\psi_2}{d\sigma}, \quad \frac{d\vartheta}{d\sigma} = \frac{T\ell}{rfL}\frac{d\psi_2}{d\sigma}, \quad \frac{d\pi_N}{d\sigma} = -\ell\frac{d\psi_2}{d\sigma}, \tag{71}$$

$$\xi\omega_N = -\frac{\ell^2}{rf^2L}\psi_2^2, \quad \xi\vartheta = -\frac{T\ell^2}{rf^2L^2}\psi_2^2, \quad \omega_\alpha\pi^\alpha = \frac{1}{rf}\psi_2^2, \quad \vartheta\omega_N = \frac{T\ell^2}{r^2f^2L}\psi_2^2, \tag{72}$$

$$\omega_N \pi_N = -\frac{\ell^2}{rf} \psi_2^2, \quad \xi \pi_N = \frac{\ell^2}{fL} \psi_2^2, \quad \vartheta \pi_N = -\frac{T\ell^2}{rfL} \psi_2^2, \quad \omega_\alpha \omega^\alpha = -\frac{1}{r^2 f^2} \psi_2^2. \tag{73}$$

Using (70)-(73) into system (53), the following transport equation for the amplitude ψ_2 can be deduced:

$$\frac{d\psi_2}{d\sigma} + \frac{L}{\ell} \Omega \psi_2 + \frac{rfL^2}{2T\chi\ell} \psi_2 - Q \psi_2^2 = 0, \tag{74}$$

where

$$Q = \frac{1}{2} \left[\frac{7(1 - krTf)}{rf} - \frac{2kT}{2krTf - 1} + 3T(k_r r - k_T T) \right] L^2.$$

By virtue of relation (62), equation (74) writes as

$$\frac{d\psi_2}{d\tau} + \frac{L}{\ell^2} \Omega \psi_2 + \frac{rfL^2}{2T\chi\ell^2} \psi_2 - \frac{Q}{\ell} \psi_2^2 = 0, \tag{75}$$

where

$$\frac{1}{\ell^2} = \frac{2(krTf - 1)}{2krTf - 1} \Rightarrow \frac{1}{\ell} = \sqrt{\frac{2(krTf - 1)}{2krTf - 1}}.$$

Analogously to the previous case, if we put

$$\Lambda_0 = \frac{rf}{2T\chi} \frac{L^2}{\ell^2} = \frac{rf}{2T\chi(2krTf - 1)}, \quad a_0 = -\frac{L}{\ell^2} = -\frac{2(krTf - 1)}{2krTf - 1} L, \quad P_0 = -\frac{Q}{\ell},$$

the transport equation for the amplitude ψ_2 can be written as

$$\frac{d\psi_2}{d\tau} + (\Lambda_0 - a_0 \Omega) \psi_2 + P_0 \psi_2^2 = 0. \tag{76}$$

Let us underline how nicely this equation shows the interplay of damping and steepening tendencies in the linear damping term $(\Lambda_0 - a_0 \Omega)$ and the nonlinear steepening term $P_0 \psi_2^2$.

This equation can be integrated to yield

$$\psi_2 = \frac{\psi_{02} \exp(-\Lambda_0 \tau) I_1(\tau)}{1 + \psi_{02} P_0 I_2(\tau)}, \tag{77}$$

where ψ_{02} is the value of ψ_2 on the wave front at $\tau = 0$,

$$I_1(\tau) = \exp\left(a_0 \int_0^\tau \Omega(\hat{\tau}) d\hat{\tau}\right)$$

and

$$I_2(\tau) = \int_0^\tau I_1(\hat{\tau}) \exp(-\Lambda_0 \hat{\tau}) d\hat{\tau}.$$

Introducing the initial principal curvatures k_{01} and k_{02} and using (65) where $\Omega_0 = \frac{1}{2}(k_{01} + k_{02})$ and $k_0 = k_{01} k_{02}$, we find that

$$I_1(\tau) = \{(1 - k_{01} a_0 \tau)(1 - k_{02} a_0 \tau)\}^{-1/2} \tag{78}$$

and

$$I_2(\tau) = \int_0^\tau \frac{\exp(-\Lambda_0 \hat{\tau})}{\sqrt{(1 - k_{01} a_0 \hat{\tau})(1 - k_{02} a_0 \hat{\tau})}} d\hat{\tau}. \tag{79}$$

We shall now discuss the solution (77) under the assumption that both k_{01} and k_{02} are nonpositive which corresponds to the case of diverging waves.

– *Diverging waves*

In this case it is easy to verify that $I_1(\tau)$ and $I_2(\tau)$ (in the form (78) and (79)) both converge as $\tau \rightarrow \infty$.

The denominator of (77) can be written as

$$1 + \frac{\psi_{02}}{\psi_c} \left\{ 1 - \frac{\int_{\tau}^{\infty} F(\hat{\tau}) d\hat{\tau}}{\int_0^{\infty} F(\hat{\tau}) d\hat{\tau}} \right\}, \tag{80}$$

in which the quantity ψ_c defined by

$$\psi_c = \left\{ P_0 \int_0^{\infty} F(\hat{\tau}) d\hat{\tau} \right\}^{-1}, \tag{81}$$

where $F(\tau) = \{(1 - k_{01}a_0\tau)(1 - k_{02}a_0\tau)\}^{-1/2} \exp(-\Lambda_0\tau)$, is finite and its sign depends on P_0 .

Let us observe that the expression in brace brackets in (80) increases monotonically from 0 to 1 as τ increases from 0 to ∞ .

We can thus conclude that:

- If $\psi_{02} > 0$ (*i.e.* an expansive wave front), and $P_0 > 0$ (so ψ_c is also positive), it follows from (77) that $\psi_2 \rightarrow 0$ as $\tau \rightarrow \infty$ and the discontinuities damp out.
- If $\psi_{02} > 0$ and $P_0 < 0$ (so ψ_c is negative), we must distinguish two cases:
 - If $|\psi_c| < \psi_{02}$ it follows from (77) that the denominator will vanish at a finite time $\tau_c > 0$ given by

$$\int_{\tau_c}^{\infty} F(\hat{\tau}) d\hat{\tau} = \left(1 - \frac{|\psi_c|}{\psi_{02}} \right) \int_0^{\infty} F(\tau) d\tau. \tag{82}$$

Hence, from (77) it follows that $\psi_2 \rightarrow \infty$ as $\tau \rightarrow \tau_c$, *i.e.* the wave front steepens into a shock in finite time τ_c .

- If $|\psi_c| > \psi_{02}$ the wave decays since from (77), $\psi_2 \rightarrow 0$ as $\tau \rightarrow \infty$.
- If $\psi_{02} < 0$ (*i.e.* a compressive wave front) and $P_0 < 0$, it follows from (77) that $|\psi_2| \rightarrow 0$ as $\tau \rightarrow \infty$.
- If $\psi_{02} < 0$ and $P_0 > 0$, we must distinguish two cases:
 - If $|\psi_{02}| < \psi_c$, the amplitude decays, since from (77), $|\psi_2| \rightarrow 0$ as $\tau \rightarrow \infty$.
 - Conversely, if $|\psi_{02}| > \psi_c$, then the denominator of (77) will vanish at a finite time τ_c given by

$$\int_{\tau_c}^{\infty} F(\hat{\tau}) d\hat{\tau} = \left(1 - \frac{\psi_c}{|\psi_{02}|} \right) \int_0^{\infty} F(\tau) d\tau. \tag{83}$$

Hence $|\psi_2| \rightarrow \infty$ as $\tau \rightarrow \tau_c$, *i.e.* the wave front steepens into a shock wave in finite time τ_c .

Thus ψ_c can be seen as critical value of the initial discontinuity in the sense that all waves with initial discontinuity less than this value attenuate, while all compressive waves with initial discontinuity greater than this value evolve into shock waves in finite time.

Relations (81) and (82)-(83) can be specialized for waves with plane, cylindrical and spherical geometry. In particular, the critical value of the initial discontinuity and the critical time are given by the following relations:

- *Plane wave.* For a plane wave front $k_{01} = k_{02} = 0$ so,

$$\psi_c = \frac{\Lambda_0}{P_0},$$

$$\tau_c = \frac{1}{\Lambda_0} \ln \left(1 - \left| \frac{\psi_c}{\psi_{02}} \right| \right)^{-1}.$$

- *Cylindrical wave.* If the outward travelling discontinuity surface is a cylinder of radius R_0 at time $\tau = 0$, then, at any time $\tau > \tau_0$, the radius of the cylinder is given by $R = R_0 + a_0\tau$. In this case $k_{01} = -1/R_0$ and $k_{02} = 0$, so

$$\psi_c = \frac{1}{P} \sqrt{\frac{\Lambda_0 a_0}{\pi R_0}} \frac{\exp(-\Lambda_0 R_0/a_0)}{\operatorname{erfc} \sqrt{\Lambda_0 R_0/a_0}},$$

$$\operatorname{erfc} \sqrt{\frac{\Lambda_0 R_0}{a_0} + \Lambda_0 \tau_c} = \left(1 - \left| \frac{\psi_c}{\psi_{02}} \right| \right) \operatorname{erfc} \sqrt{\frac{\Lambda_0 R_0}{a_0}}.$$

- *Spherical wave.* If the outward travelling discontinuity surface is a sphere of radius R_0 at time $\tau = 0$, then at any time $\tau > \tau_0$, the radius of the sphere is given by $R = R_0 + a_0\tau$. In this case $k_{01} = k_{02} = -1/R_0$, so

$$\psi_c = \frac{1}{P_0} \frac{a_0}{R_0} \frac{\exp(-\Lambda_0 R_0/a_0)}{E_i(\Lambda_0 R_0/a_0)},$$

$$E_i \left(\frac{\Lambda_0 R_0}{a_0} + \Lambda_0 \tau_c \right) = \left(1 - \left| \frac{\psi_c}{\psi_{02}} \right| \right) E_i \left(\frac{\Lambda_0 R_0}{a_0} \right).$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt, \quad E_i(x) = \int_x^\infty t^{-1} e^{-t} dt,$$

are, respectively, the complementary error function and the exponential integral.

If $|\psi_{02}| = -|\psi_c|$, it is clear from equation (77) that the wave can neither evolve into a shock, nor damp out.

7. Conclusions

A thorough discussion of the relativistic dynamics of fluids includes a number of dissipative processes [38]. The effects of internal dissipation in fluids - viscosity and thermal conductivity - are well modelled by a generalization of the basic theory called Navier-Stokes equations. Unfortunately, first approaches to constructing relativistic generalizations of the Navier-Stokes equations result in rather pathological theories [1, 2]. These theories are non-causal and without a well-posed initial value formulation (see for example [21]). Less straightforward approaches have succeeded in producing a class of causal dissipative relativistic fluid theories (e.g. Israel [6], Israel and Stewart [7], Ruggeri and Strumia [39], Pavon, Jou and Casa-Vasquez [40], Hiscock and Lindblom [21, 22], Galipò [11, 12], Hiscock and Olson [23], Geroch and Lindblom [41], Carter [31], Müller and Ruggeri [10], Muronga [42, 13], Maartens [43]).

Almost all formulations used Eckart scheme [1] or Landau-Lifshitz approach [2] for problem of heat conduction.

It is well known that for some interesting applications in a number of contexts RHIC and LHC [46, 13, 44, 45] are important to develop a robust model of general dissipative processes [4, 38, 47, 43].

Purpose of authors, in light of some results due to Silva et al. [48], Heinz et al. [49], Ván and Biró [50], Ván [51], Muronga [42], Maartens [43], is to form a generic causal theory (heat-conducting, viscous, particle-creating) which is valid both in Eckart and Landau frames.

In a first work [24] we developed a second-order theory for relativistic fluid with thermal conduction and examine the propagation of weak discontinuities in the special case of ultrarelativistic fluids in Landau-Lifshitz scheme.

In a second paper (the present), similar study is done in Eckart formulation.

In a forthcoming work, results will be compared and the two schemes will be generalized in order to consider the hypothesis (found in Carter [31] and recently picked up by Andersson and Comer [20]) of state of equilibrium in which particle current and entropy flux are not aligned.

Moreover, hypothesis about relation between bulk viscosity and matter creation, found in [52, 53, 54], will be detailed. Finally, we will land studies to arrive in the realization of that work that permit applications in RHIC and LHC [46, 42].

Acknowledgements

Work supported by G.N.F.M of I.N.d.A.M., by Tirrenoambiente s.p.a. of Messina and by research grants of the University of Messina.

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Communicated 30 November 2011; published online 8 June 2012

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