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ON ACTIONS OF THE ADDITIVE GROUP ON THE WEYL ALGEBRA

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ABSTRACT. We construct classes of examples of $H-$coactions and $H-$Galois coactions on the Weyl algebra in characteristic $p > 0$ where H is the function algebra of the additive group of dimension p^2. In particular we show that the main result obtained in a preceeding paper [G. Restuccia and H.-J. Schneider, *J. Algebra* **261**, 229 (2003)] cannot be generalized to non-commutative algebras.

Introduction

Let k be a field of characteristic $p > 0$, and H a commutative Hopf algebra with underlying algebra

$$H = k[X_1, \dots, X_n]/(X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}}),\ n \geq 1, s_1 \geq \dots \geq s_n \geq 1.$$

Here, $k[X_1, \dots, X_n]$ denotes the commutative polynomial algebra in n indeterminates $X_1, \dots, X_n$. For all i let x_i be the residue class of X_i in H. Then the elements $x_1^{t_1} \cdots x_n^{t_n}$ for $0 \leq t_i \leq p^{s_i} - 1, 1 \leq i \leq n$ are a basis of H. Let A be a right H-comodule algebra with structure map $\delta\colon A \to A \otimes H$. Then for all $a \in A$,

$$\delta(a) = a \otimes 1 + \sum_{i=1}^{n} D_i(x) \otimes x_i + \text{ terms of higher order in the } x_i's,$$

where $D_i\colon A \to A, 1 \leq i \leq n$ are derivations of A.

Let $R = A^{\mathrm{co}H} = \{a \in A \mid \delta(a) = a \otimes 1\}$ be the subalgebra of H-coinvariant elements. The extension $R \subset A$ is called an *H-Galois extension* or A is H-Galois if the Galois map

$$A \otimes_R A \to A \otimes H,\ a \otimes b \mapsto a\delta(b),$$

is bijective.

The quotient algebra $Q = H/(x^p \mid x \in H^+)$ is a quotient Hopf algebra of H, and it represents the first Frobenius kernel of the group scheme $\mathrm{Sp}(H)$ [1, Chap.II, §7]. Let $\pi : H \to Q$ be the quotient map. We identify

$$Q \cong k[Y_1, \dots, Y_n]/(Y_1^p, \dots, Y_n^p), \pi(x_i) \mapsto y_i = \text{ residue class of } Y_i.$$

Let $B = A^{\mathrm{co}Q} = \{a \in A \mid (\mathrm{id} \otimes \pi)(\delta(a)) = a \otimes 1\}$ be the algebra of Q-coinvariant elements of the induced coaction

$$\delta_Q : A \xrightarrow{\delta} A \otimes H \xrightarrow{\mathrm{id}\otimes\pi} A \otimes Q.$$

Assume that A is a commutative algebra. Then we proved in [2, Theorem 4.1] a Jacobi criterion for $R \subset A$ to be H-Galois which can be formulated in terms of the derivations D_i. In particular we showed in the commutative case that $R \subset A$ is H-Galois if $B \subset A$ is Q-Galois.

For arbitrary algebras it is therefore natural to ask

Question 1. Assume that $B \subset A$ is Q-Galois. Is $R \subset A$ H-Galois?

The additive group is given by $H_a = H$ as an algebra and with Hopf algebra structure defined by $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$, for all $1 \leq i \leq n$. For H_a-actions and arbitrary algebras A we showed in [2, Theorem 3.1] that $R \subset A$ is H_a-Galois and A is faithfully flat as a (left or right) R-module if and only if there are elements $a_1, \dots a_n \in A$ with

$$\delta(a_i) = a_i \otimes 1 + 1 \otimes x_i \text{ for all } 1 \leq i \leq n.$$

Let $K = k[H^p]$. Then K is a normal Hopf subalgebra of H, $H/HK^+ = Q$, and δ defines by restriction a K-comodule algebra structure on B with $R = B^{\mathrm{co}K}$. Assume that $R \subset B$ is faithfully flat K-Galois, and $B \subset A$ is faithfully flat Q-Galois. Then $R \subset A$ is faithfully flat H-Galois by [3, Theorem 4.5.1]. However the argument in [3, Theorem 4.5.1] is not constructive.

Thus we might ask for actions of the additive group where we assume $s_1 \geq \dots \geq s_m > 1, s_{m+1} = \dots s_1 = 1, 1 \leq m \leq n$,

Question 2. Let $H = H_a$, and assume that we are given elements $b_1, \dots, b_m \in B$ and $c_1, \dots, c_n \in A$ with

$$\delta(b_j) = b_j \otimes 1 + 1 \otimes x_j^p, \text{ for all } 1 \leq j \leq m,$$
$$\delta_Q(c_i) = c_i \otimes 1 + 1 \otimes y_i, \text{ for all } 1 \leq i \leq n.$$

Is there a constructive way to find elements $a_1, \dots, a_n \in A$ with

$$\delta(a_i) = a_i \otimes 1 + 1 \otimes x_i \text{ for all } 1 \leq i \leq n?$$

In the following we consider actions of the additive group with $n = 1, s_1 = 2$, that is

$$H = k[X]/(X^{p^2}) \cong k[x],\ x^{p^2} = 0,\ \Delta(x) = x \otimes 1 + 1 \otimes x,$$

on the Weyl algebra

$$A_1 = k\langle u, v \mid vu = uv + 1\rangle.$$

In this situation we give a negative answer to question 1, and a partial positive answer to question 2. In particular, we construct classes of examples of H-coactions and of H-Galois coactions on the Weyl algebra A_1.

1. The main result

Recall that $H = k[x]$ with $x^{p^2} = 0, \Delta(x) = x\otimes 1 + 1 \otimes x$, and k-basis $x^i, 0 \leq i \leq p^2 - 1$. The first Lemma is well-known. We sketch a proof for completeness.

Lemma 1.1. *Let A be an algebra, and $\delta : A \to A \otimes k[x]$ a linear map. For all $a \in A$, write*

$$\delta(a) = \sum_{i=0}^{p^2-1} D_i(a) \otimes x^i,$$

where each $D_i : A \to A$ is a linear map. Then δ is a comodule algebra structure if and only if

(1) $D_i(ab) = \sum_{j=0}^{i} D_j(a)D_{i-j}(b)$ *for all* $a, b \in A, 0 \le i < p^2$.
(2) $D_i(1) = 0$ *for all* $1 \le i < p^2,\ D_0 = \mathrm{id}$.
(3) $D_i = \frac{D_1^{i_0}}{i_0!}\frac{D_p^{i_1}}{i_1!}$, *for all* $0 \le i_0, i_1 < p, i = i_0 + i_1 p$.
(4) $D_1 D_p = D_p D_1, D_1^p = 0, D_p^p = 0$.

Proof. It is easy to see that δ is a comodule algebra structure if and only if (1) and (2) hold, and $D_r D_s = \begin{cases} \binom{r+s}{r} D_{r+s}, & \text{if } r+s \le p^2-1 \\ 0, & \text{if } r+s \ge p^2. \end{cases}$ The Lemma follows from the identities $\binom{kp}{p} \equiv k \bmod p$ for all $k \ge 1$, and $\binom{k+lp}{k} \equiv 1 \bmod p$, for all $0 \le k, l < p$. □

For $k[x]$-coactions δ we use the notation of Lemma **1.1**. In particular, if $a \in A$ with $D_1(a) = 1$, then we can write

$$\delta(a) = a \otimes 1 + 1 \otimes x + \sum_{i=1}^{p-1} \frac{D_p^i(a)}{i!} \otimes x^{ip}. \tag{1.1}$$

Lemma 1.2. *Let A be an algebra, and $\delta : A \to A \otimes k[x]$ a $k[x]$-comodule algebra structure. Let $a \in A$ with $D_1(a) = 1$. Then*

$$D_p(a^p) = 1 + \sum_{i=0}^{p-1} a^i D_p(a) a^{p-i-1},$$

and if a^p is central in A, then $\delta(a^{p^2}) = a^{p^2} \otimes 1$.

Proof. We collect all terms of the form $b \otimes x^p, b \in A$, in

$$(\delta(a))^p = (a \otimes 1 + 1 \otimes x + D_p(a) \otimes x^p + \sum_{i=2}^{p-1} \frac{D_p^i(a)}{i!} \otimes x^{ip})^p.$$

The contribution of products of p summands of the form $a \otimes 1$ or $1 \otimes x$ is $1 \otimes x^p$, since $(a \otimes 1 + 1 \otimes x)^p = a^p \otimes 1 + 1 \otimes x^p$. If we take products with at least one factor $D_p(a) \otimes x^p$, all the other factors must be $a \otimes 1$ (or the product is 0). This proves the formula for $D_p(a^p)$.

If a^p commutes with $D_p(a)$, then we compute $\delta(a)^p$ by the binomial formula:

$$(a \otimes 1 + 1 \otimes x + (\sum_{i=1}^{p-1} \frac{D_p^i(a)}{i!} \otimes x^{(i-1)p})(1 \otimes x^p))^p = a^p \otimes 1 + 1 \otimes x^p.$$

Thus $\delta(a^{p^2}) = a^{p^2} \otimes 1$. □

Remark 1.3. We note some properties of the Weyl algebra A_1.

(1) The general commutation rule

$$v^s u^t = \sum_{i=0}^{\mathrm{Min}(s,t)} u^{t-i} v^{s-i} \binom{s}{i} t(t-1)\cdots(t-i+1).$$

follows by induction on s, t.

(2) In particular, v^p and u^p are central in A_1.
(3) Since A_1 is an Ore extension, A_1 is an integral domain, and the elements $u^i v^j, i, j \geq 0$, form a k-basis of A_1.
(4) The only units in A_1 are the elements in k. This follows from (1) using (3).

We need the following identity in the Weyl algebra.

Lemma 1.4. $\sum_{s=0}^{p-1} v^s u^t v^{p-s-1} = \begin{cases} -u^{(k-1)p}, & \textit{if } t = kp-1, k \geq 1 \\ 0, & \textit{if } t \not\equiv -1 \bmod p. \end{cases}$

Proof. By Remark **1.3** (1),

$$\begin{aligned}\sum_{s=0}^{p-1} v^s u^t v^{p-s-1} &= \sum_{s=0}^{p-1}\sum_{i=0}^{s} \binom{s}{i} t(t-1)\cdots(t-i+1)u^{t-i}v^{s-i}v^{p-s-1} \\ &= \sum_{l=0}^{p-1}\sum_{s=i}^{p-1} \binom{s}{i} t(t-1)\cdots(t-i+1)u^{t-i}v^{p-i-1}.\end{aligned}$$

The identity

$$(\sum_{s=0}^{p-1}(X+1)^s)X = (X+1)^p - 1 = X^p$$

in the polynomial algebra over $\mathbb{Z}/(p)$ implies

$$\sum_{s=0}^{p-1}\sum_{li=0}^{s} \binom{s}{i} X^i = \sum_{i=0}^{p-1}\sum_{s=i}^{p-1} \binom{s}{i} X^i = X^{p-1}, \text{ hence}$$

$$\sum_{s=i}^{p-1} \binom{s}{i} \equiv 0 \bmod p \text{ for all } 0 \leq i \leq p-2.$$

Thus

$$\begin{aligned}\sum_{s=0}^{p-1} v^s u^t v^{p-s-1} &= t(t-1)\cdots(t-(p-1)+1)u^{t-(p-1)} \\ &= \begin{cases} -u^{(k-1)p}, & \text{if } t = kp-1, k \geq 1 \\ 0, & \text{if } t \not\equiv -1 \bmod p, \end{cases}\end{aligned}$$

since $t(t-1)\cdots(t-p+2) \equiv \begin{cases} -1, & \text{if } t \equiv -1 \bmod p \\ 0, & \text{if } t \not\equiv -1 \bmod p. \end{cases}$ □

Note that $K = k[H^p] = k[x^p]$, and the quotient map $H \to H/HK^+$ can be identified with $\pi : k[x] \to k[y], x \mapsto y$, where $k[y] = k[Y]/(Y^p)$, y the residue class of Y, and $\Delta(y) = y \otimes 1 + 1 \otimes y$.

Lemma 1.5. *Let $\delta : A_1 \to A_1 \otimes k[x]$ be a $k[x]$-comodule algebra structure. Assume that $\delta(u) = u \otimes 1$, and $D_1(v) = 1$. Then*

(1) $D_1(bv_i) = biv^{i-1}$, *and* $D_p(bv^{ip}) = biv^{(i-1)p}D_p(v^p)$ *for all* $b \in k[u], i \geq 0$.

(2) $A^{\mathrm{cok}[y]} = \{a \in A \mid D_1(a) = 0\} = k[u, v^p]$.

(3) $A^{\mathrm{cok}[x]} = \begin{cases} k[u, v^{p^2}], & \textit{if } D_p(v^p) \neq 0, \\ k[u, v^p], & \textit{if } D_p(v^p) = 0. \end{cases}$

(4) *There are* $a_i \in k[v^p], i \geq 0$, *with* $D_p(v) = \sum_{t \geq 0} a_t u^t$, *and*

$$D_p(v^p) = 1 - \sum_{k \geq 1} a_{kp-1} u^{(k-1)p}.$$

Proof. (1) By Lemma **1.1** (1), D_1 is a derivation, and the formula for $D_p(bv^{ip})$ follows by induction on i from Lemma **1.1**.

(2) $A^{\mathrm{cok}[y]} = \{a \in A \mid D_1(a) = 0\}$ is clear from the definition, and $\{a \in A \mid D_1(a) = 0\} = k[u, v^p]$ follows from (1).

(3) By (2), $A_1^{\mathrm{cok}[x]} = \{a \in k[u, v^p] \mid D_p(a) = 0\}$. Let $a = \sum_{i \geq 0} a_i v^{ip} \in k[u, v^p], a_i \in k[u], i \geq 0$. Then $D_p(a) = \sum_{i \geq 0} a_i i v^{(i-1)p} D_p(v^p)$ by (2). This implies (3) since A_1 is an integral domain.

(4) By Lemma **1.1**, $D_1 D_p(v) = D_p D_1(v) = D_p(1) = 0$, hence we can write $D_p(v) = \sum_{t \geq 0} a_t u^t, a_t \in k[v^p], t \geq 0$. Since the elements a_t are central in A_1, it follows from Lemma **1.2** that

$$D_p(v^p) = 1 + \sum_{s=0}^{p-1} v^s (\sum_{t \geq 0} a_t u^t) v^{p-s-1} = 1 + \sum_{t \geq 0} a_t \sum_{s=0}^{p-1} v^s u^t v^{p-s-1},$$

and the claim follows from Lemma **1.4**. □

Our main result is

Theorem 1.6. *Let* $A_1 = k\langle u, v \mid vu = uv + 1\rangle$, *and* $\delta : A_1 \to A_1 \otimes k[x]$ *a comodule algebra structure with* $B = A_1^{\mathrm{cok}[y]}$. *Assume that* $\delta(u) = u \otimes 1$, *and* $D_1(v) = 1$. *Then* $B = k[u, v^p]$, A_1 *is* $k[y]$*-Galois, and there are elements* $a_t \in k[v^p], t \geq 0$, *with* $D_p(v) = \sum_{t \geq 0} a_t u^t$, *and the following are equivalent:*

(1) A_1 *is* $k[x]$*-Galois.*
(2) *There is an element* $a \in A_1$ *with* $\delta(a) = a \otimes 1 + 1 \otimes x$.
(3) $0 \neq D_p(v^p) \in k$.
(4) $1 \neq a_{p-1} \in k$, *and* $a_{kp-1} = 0$ *for all* $k \geq 2$.
(5) B *is* $k[x^p]$*-Galois.*
(6) *There is an element* $b \in A_1$ *with* $\delta(b) = b \otimes 1 + 1 \otimes x^p$.

Proof. By Lemma **1.5** (2), (3), $B = k[u, v^p]$ and $A^{\mathrm{cok}[x]}$ are commutative. Hence (1) ⇔ (2), and (5) ⇔ (6) by [2, Theorem 3.1]. Note that in (6), $b \in B$, since $B = \{a \in A_1 \mid D_1(a) = 0\}$.

A_1 is $k[y]$-Galois since $\delta_{k[y]}(v) = v \otimes 1 + 1 \otimes y$.

The existence of the elements a_t is shown in Lemma **1.5** (4).

(3) ⇔ (4) follows from Lemma (**1.5**) (4).

(2) $\Rightarrow$ (3): Since $D_1(v) = 1$, it follows from Lemma **1.5** (2) and (1) that there are $b_i \in k[u], i \geq 0$, with $a = v + \sum_{i\geq 0} b_i v^{ip}$, and

$$0 = D_p(v) + \sum_{i\geq 0} b_i i v^{(i-1)p} D_p(v^p). \tag{1.2}$$

By Lemma **1.5** (4) there are $a_t \in k[v^p], t \geq 0$, with

$$D_p(v) = \sum_{t\not\equiv -1 \bmod p} a_t u^t + (\sum_{k\geq 1} a_{kp-1} u^{(k-1)p}) u^{p-1}, \tag{1.3}$$

$$D_p(v^p) = 1 - \sum_{k\geq 1} a_{kp-1} u^{(k-1)p}. \tag{1.4}$$

Since $k[u^p, v^p] \subset k[u, v^p]$ is a free ring extension with $k[u^p, v^p]$-basis $1, u, \ldots, u^{p-1}$ we can find $c_j \in k[u^p, v^p], j \geq p-1$, with

$$\sum_{i\geq 0} b_i i v^{(i-1)p} = \sum_{j=0}^{p-1} c_j u^j. \tag{1.5}$$

Hence we obtain from **(1.2)**, **(1.3)**, **(1.4)** and **(1.5)**

$$0 = \sum_{t\not\equiv -1 \bmod p} a_t u^t + (\sum_{k\geq 1} a_{kp-1} u^{(k-1)p}) u^{p-1} + \sum_{j=0}^{p-1} c_j u^j (1 - \sum_{k\geq 1} a_{kp-1} u^{(k-1)p}).$$

The coefficient of u^{p-1} in the $k[u^p, v^p]$-basis representation of the expression on the right hand side is

$$0 = \sum_{k\geq 1} a_{kp-1} u^{(k-1)p} + c_{p-1}(1 - \sum_{k\geq 1} a_{kp-1} u^{(k-1)p}).$$

Let $F = \sum_{k\geq 1} a_{kp-1} u^{(k-1)p}$, and $c = c_{p-1}$. Then $0 = F + c(1-F)$, hence $c = F(c-1)$, and $0 = F(1+(c-1)(1-F))$. Since $k[u^p, v^p]$ is a polynomial ring in the variables u^p, v^p, it follows that $F = 0$ or $1 - F$ is a nonzero scalar in k. Thus $0 \neq 1 - F = D_p(v_p) \in k$.

(3) $\Rightarrow$ (2): Let $\alpha = D_p(v^p)$, hence $0 \neq \alpha \in k$, by (3). By Lemma **1.5** (2), $D_p(v) \in k[u, v^p]$, since $D_1 D_p(v) = D_p D_1(v) = 0$. Hence there are $c_i \in k[u], i \geq 0$, with $D_p(v) = \sum_{i\geq 0} c_i v^{ip}$. By Lemma **1.1** (4), $D_p^p(v) = 0$. Thus we get from Lemma **1.5** (1)

$$0 = D_p^p(v) = \sum_{i\geq 0} c_i i(i-1)\cdots(i-(p-2)) v^{(i-(p-1))p} \alpha^{p-1}.$$

Since for all $i = kp-1, k \geq 1$, $i(i-1)\cdots(i-(p-2)) \equiv (p-1)! \equiv -1 \bmod p$, and $i - (p-1) = (k-1)p \geq 0$, we see that $c_i = 0$ for all $i \equiv -1 \bmod p$.

Then

$$a = v - \sum_{\substack{i\geq 1 \\ i\not\equiv 0 \bmod p}} \frac{c_{i-1}}{i\alpha} v^{ip} \tag{1.6}$$

is the element we want, since by Lemma **1.5** (1), $D_1(a) = 1$, and

$$D_p(a) = \sum_{i\geq 0} c_i v^{ip} - \sum_{\substack{i\geq 1 \\ i\not\equiv 0 \bmod p}} \frac{c_{i-1}}{i\alpha} i v^{(i-1)p} \alpha = 0,$$

hence $\delta(a) = a \otimes 1 + 1 \otimes x$ by Lemma **1.1**.

(6) $\Rightarrow$ (3): Let $b_i \in k[u], i \geq 0$ with $b = \sum_{i\geq 0} b_i v^{ip}$. Then by Lemma **1.5** (1),

$$1 = D_p(b) = \sum_{i\geq 0} b_i i v^{(i-1)p} D_p(v^p).$$

Hence $0 \neq D_p(v^p) \in k$, since the only invertible elements in the Weyl algebra A_1 are scalars.

(3) $\Rightarrow$ (6): Let $\alpha = D_p(v^p)$. Then $0 \neq \alpha \in k$, and $b = \alpha^{-1} v^p$ satisfies (6). $\square$

The implication (6) $\Rightarrow$ (2) also follows from the abstract argument in [3, Theorem 4.5.1], since $B \subset A_1$ is a faithfully flat $k[y]$-Galois extension. In our proof of Theorem **1.6** however, we constructed the element a explicitly.

2. Examples

Our first class of examples shows that Question 1 in the introduction has a negative answer.

Example 2.1. Let $a_t \in k[v^{p^2}], t \geq 0$, and $b = \sum_{t\geq 0} a_t u^t \in k[u, v^{p^2}]$. Then there is a $k[x]$-comodule algebra structure $\delta : A_1 \to A_1 \otimes k[x]$ with

$$\delta(u) = u \otimes 1, \delta(v) = v \otimes 1 + 1 \otimes x + b \otimes x^p,$$

and

(1) $A_1^{\mathrm{co} k[x]} = \begin{cases} k[u, v^{p^2}], & \text{if } \sum_{k\geq 1} a_{kp-1} u^{(k-1)p} \neq 1, \\ k[u, v^p], & \text{if } \sum_{k\geq 1} a_{kp-1} u^{(k-1)p} = 1. \end{cases}$

(2) A_1 is $k[y]$-Galois.

(3) A_1 is $k[x]$-Galois if and only if $1 \neq a_{p-1} \in k$, and $a_{kp-1} = 0$ for all $k \geq 2$.

Moreover, if A_1 is $k[x]$-Galois, and if we write $b = \sum_{i\geq 0} c_i v^{ip}$, with $c_i \in k[u], i \geq 0$, then

$$\delta(a) = a \otimes 1 + 1 \otimes x, \text{ where } a = v - \sum_{\substack{i\geq 1 \\ i \not\equiv 0 \bmod p}} \frac{c_{i-1}}{i(1 - a_{p-1})} v^{ip}.$$

Proof. Since $ub = bu$, δ defines an algebra map. It is easy to check that δ is coassociative, that is $(\mathrm{id} \otimes \Delta)(\delta(v)) = (\delta \otimes \mathrm{id})(\delta(v))$, since $\delta(v^{p^2}) = v^{p^2} \otimes 1$ by Lemma **1.2**, and hence $\delta(b) = b \otimes 1$. (1), (2), and (3) follow from Theorem **1.6**. The formula for a is a special case of (**1.6**). $\square$

Example 2.2. Let $c_0, \ldots, c_{p-2} \in k[u^p, v^{p^2}]$, and define

$$b_i = \sum_{j=0}^{p-1-i} \binom{i+j-1}{j} \frac{1}{i} c_{i+j-1} v^{jp}, 1 \leq i \leq p-1.$$

Then there is a $k[x]$-comodule algebra structure $\delta : A_1 \to A_1 \otimes k[x]$ with

$$\delta(u) = u \otimes 1, \delta(v) = v \otimes 1 + 1 \otimes x + \sum_{i=1}^{p-1} b_i \otimes x^{ip},$$

and A_1 is $k[x]$-Galois. Moreover,

$$\delta(a) = a \otimes 1 + 1 \otimes x, \text{ where } a = v - \sum_{i=1}^{p-1} \frac{c_{i-1}}{i} v^{ip}.$$

Proof. Since $ub_i = b_i u$ for all $1 \leq i \leq p-1$, δ defines an algebra map. To check coassociativity, we compute

$$\begin{aligned}(\delta \otimes \mathrm{id})(\delta(v)) &= v \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + \sum_{i=1}^{p-1} b_i \otimes x^{ip} \otimes 1 \\ &\quad + 1 \otimes 1 \otimes x + \sum_{i=1}^{p-1} \delta(b_i) \otimes x^{ip}, \\ (\mathrm{id} \otimes \Delta)(\delta(v)) &= v \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x \\ &\quad + \sum_{i=1}^{p-1} b_i \otimes \sum_{j=0}^{i} \binom{i}{j} x^{jp} \otimes x^{(i-j)p}.\end{aligned}$$

Hence $(\delta \otimes \mathrm{id})(\delta(v)) = (\mathrm{id} \otimes \Delta)(\delta(v))$ if and only if

$$\textbf{(2.1)} \qquad \delta(b_i) = \sum_{j=0}^{p-1-i} \binom{i+j}{j} b_{i+j} \otimes x^{jp}, \text{ for all } 1 \leq i \leq p-1.$$

Since $vb_i = b_i v$ for all $1 \leq i \leq p-1$, we have $\delta(v^p) = v^p \otimes 1 + 1 \otimes x^p$. Hence for all $1 \leq i \leq p-1$,

$$\begin{aligned}\delta(b_i) &= \sum_{j=0}^{p-1-i} \binom{i+j-1}{j} \frac{1}{i} c_{i+j-1} \Big(\sum_{k=0}^{j} \binom{j}{k} v^{(j-k)p} \otimes x^{kp}\Big) \\ &= \sum_{k=0}^{p-1-i} \frac{1}{i} \sum_{l=0}^{p-1-(i+k)} \binom{i+k+l-1}{k+l} c_{i+k+l-1} \binom{k+l}{k} v^{lp} \otimes x^{kp}.\end{aligned}$$

On the other hand, for all $1 \leq i \leq p-1$,

$$\begin{aligned}&\sum_{k=0}^{p-1-i} \binom{i+k}{k} b_{i+k} \otimes x^{kp} = \\ &\sum_{k=0}^{p-1-i} \binom{i+k}{k} \sum_{l=0}^{p-1-(i+k)} \binom{i+k+l-1}{l} \frac{1}{i+k} c_{i+k+l-1} v^{lp} \otimes x^{kp}.\end{aligned}$$

This proves **(2.1)** since

$$\frac{1}{i} \binom{i+k+l-1}{k+l} \binom{k+l}{k} = \binom{i+k}{k} \binom{i+k+l-1}{l} \frac{1}{i+k}.$$

Thus δ is coassociative, and A_1 is $k[x]$-Galois by Theorem **1.6** since $D_p(v^p) = 1$. The formula for a is again a special case of **(1.6)**.

□

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