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NONLOCAL AND ROTATIONAL EFFECTS IN QUANTUM TURBULENCE

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ABSTRACT. We discuss phenomenological equations for the evolution of vortex tangle in counterflow superfluid turbulence, which takes into account the influence of the non local effects, both in absence and in the presence of rotation.

1. Introduction

Nonlocal terms are receiving much attention in current transport theory, due to the recent stimulus of research on nanoscale systems, where the size of the system becomes comparable to the mean-free-path of particles. We want to stress here some questions on superfluid turbulence in narrow channels [1-4], a situation which is an interesting candidate to be considered from this perspective. Furthermore, in recent years there has been increasing attention in superfluid turbulence [1-4], because of a renewed interest in the behavior of quantized vortices -in superfluids, in Bose-Einstein condensates, and in superconductors-, and because it may be of practical interest in cryogenic applications to keep small systems at low temperatures by removing heat through the flow of superfluid helium along thin capillaries [2].

Superfluid turbulence in ^{4}He [1-4] has been much investigated in two physical situations: rotating containers and counterflow experiments (an experimental situation characterized by no matter flow but only heat transport). In both cases, the vortex array is described by introducing a scalar quantity L, the average vortex line length per unit volume, briefly called *vortex line density* and whose dimensions are $(length)^{-2}$. In the first case the structure of the vortex lines is an ordered array of lines aligned along the rotation axis; in this case L equals the areal density L_R:

$$L = L_R = \frac{2\Omega}{\kappa}, \tag{1.1}$$

where κ is the quantum of vorticity, ascribed by $\kappa = h/m_4$, with h the Planck constant, and m_4 the mass of ^{4}He atom ($\kappa \simeq 9.97\ 10^{-4}$cm^2/s). Equation (1.1) is valid provided that Ω exceeds a small critical value Ω_c [5].

In counterflow experiments, the vortex line structure is a disordered tangle of lines; in this case L depends on the square of the counterflow velocity $\mathbf{V}$ and on the dimension d of the channel, $L = L(V^2, d)$, where $V = | < \mathbf{V} > |$ is the modulus of the spatial average of counterflow velocity $\mathbf{V} = \mathbf{v}_n - \mathbf{v}_s$, $\mathbf{v}_n$ and $\mathbf{v}_s$ being the velocities of the normal and superfluid components. For high values of V, one obtains the well known Gorter-Mellink

law [6]:

$$L = L_H = \gamma_H^2 V^2. \tag{1.2}$$

with γ_H a parameter dependent on temperature.

The most well known equation in the field of superfluid turbulence is Vinen's equation [7], which describes the evolution of line density L, in homogeneous counterflow turbulence. Vinen suggested that in this situation there is a balance between generation and decay processes, which leads to a steady state of quantum turbulence in the form of a self-maintained vortex tangle. The form of Vinen's equation is [7]:

$$\frac{dL}{dt} = \alpha_v V L^{3/2} - \beta_v \kappa L^2, \tag{1.3}$$

where α_v and β_v are dimensionless parameters, which depend on temperature. To derive his equation, Vinen assumes that the time derivative of L is composed of two opposite contributions

$$\frac{dL}{dt} = \left[\frac{dL}{dt}\right]_f - \left[\frac{dL}{dt}\right]_d, \tag{1.4}$$

where subscripts f and d denote formation and destruction of vortices per unit of time and volume, respectively. Vinen assumes that the term $[dL/dt]_f$ depends on the quantum of circulation κ, the local and instantaneous value of L and the intensity V of the counterflow velocity; dimensional analysis leads to the equation [7,8]:

$$\left[\frac{dL}{dt}\right]_f = V L^{3/2} \phi_f \left[\frac{V}{\kappa L^{1/2}}\right]. \tag{1.5}$$

By analogy with the growth of a vortex ring, Vinen assumed that the dimensionless function ϕ_f is constant, obtaining:

$$\left[\frac{dL}{dt}\right]_f = \alpha_v V L^{3/2}; \tag{1.6}$$

the form of the $[dL/dt]_d$ destruction term was determined in analogy with classical turbulence, obtaining:

$$\left[\frac{dL}{dt}\right]_d = -\beta_v \kappa L^2. \tag{1.7}$$

Substituting (1.6) and (1.7) in (1.4) one obtains immediately (1.3). Vinen's equation is found to be successful in the description of completely developed turbulence, but it cannot describe the full complexity of the different regimes observed experimentally [9,10]: a laminar regime, a transition to the laminar to the turbulent low-density state (TI), and a high-density state (TII) that can be associated with the homogeneous state. We will see that this limitation may be overcome when nonlocal effects and the influence of the walls are taken into account: in fact, these effects are important in the transition from laminar to turbulent regimes and in the last phases of vortex decay [11-12].

A second limitation of (1.3) is that it does not include rotational effects. The interest in combined rotation and heat flux [2,13-19] is great because it turns out, experimentally [20,13] and numerically [14-16], that both effects are not merely additive, but show an interplay between the ordered vortices of rotation and the disordered ones of counterflow. In this combined situation non local effects are important in the description of the transition

from the vortex-free region to the turbulent region and in the low-density vortex region. It would be useful to describe in a single formalism turbulence in rotating containers and in counterflow. In the present paper we will deal with nonlocal effects and the influence of the walls, and combination of rotation and counterflow.

2. Nonlocal effects and influence of the walls on superfluid turbulence

Equation (1.3) has been given a physical microscopic basis by Schwarz, starting from statistical considerations on vortex-line dynamics [21,22]. In the microscopic model by Schwarz, the vortex lines are represented in the parametric form $\mathbf{s}(\xi, t)$, ξ being the length along the line. The equation of motion of the line depends on $\mathbf{s}'$, $\mathbf{s}''$ and on the higher-order derivatives $\mathbf{s}'''$, $\mathbf{s}''''$ and so on, which follow a hierarchy of evolution equations. To truncate this hierarchy Schwarz assumed that the derivatives become uncorrelated in a distance of order of the average vortex separation $\delta \simeq L^{-1/2}$, and was able to derive Vinen's equation.

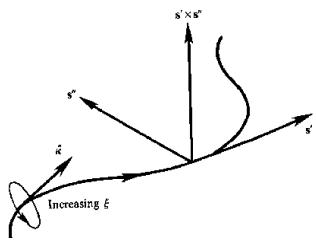

FIGURE 2.1: *Vortex line*

An open question is which would be the evolution equation for the tangle in situations where the vortex separation δ is comparable to the diameter d of the channel. This happens, for instance: (a) in the laminar and in the TI turbulent regime, (b) in the transition from TI to TII turbulent regimes, (c) in the late stages of the decay of turbulence.

Important features of vortex dynamics, not considered in the Schwarz's derivation of Vinen's equation, are the possibility of *vortex pinning* and *vortex reconnections*. Experiments have shown that pinned vortices, formed when helium was cooled through the λ-point or during previous turbulent flows, are always found in the fluid, while freely moving vortices do not live long time: either they are trapped on suitable protuberances of the wall of the container, or they lose their energy by interacting with the elementary excitations. When two vortices approach each other closely, they reconnect. Vortices can pin to the wall of the container. Pinned vortices unpin when the applied flow bends them very strongly. Schwarz's extensive numerical simulations [21,22] of vortex motion confirm these features of the dynamics of vortices.

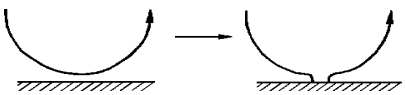

FIGURE 2.2: *A vortex pins to the wall*

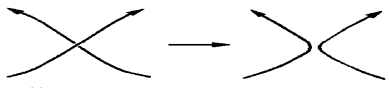

FIGURE 2.3: *Vortex reconnection*

The phenomena of vortex pinning and reconnection are important in the superfluid turbulence evolution, when the diameter d of the channel becomes comparable to the intervortex separation δ, especially in small channels. It is clear that these phenomena are typical non local effects. Nonlocal terms are needed in transport equations when the system is inhomogeneous or when its size becomes comparable to the mean-free-path of particles.

In this section, we will focus our attention on homogeneous situations in which the wall effects are important in the whole volume of the system because the mean-free-path is comparable to the size of the system. We will propose a phenomenological extension of Vinen's equation (1.3) able to describe with a single evolution equation the three stationary regimes observed in counterflow: a laminar regime at low V, the transition at the counterflow velocity V_{c1}^H from the laminar to the turbulent TI regime, the two turbulent regimes TI and TII at increasing values of V. In particular, we will consider two modifications: a) we will include non linear production terms quadratic in the counterflow velocity V, and b) we will incorporate corrections depending on δ/d, including the effects of the size of the capillary.

We suppose that both the formation and the destruction terms depend of the ratio δ/d. Being the interline space δ proportional to $L^{-1/2}$, we write:

$$\left[\frac{dL}{dt}\right]_f = \kappa L^2 \phi_f \left[\frac{V}{\kappa L^{1/2}}, \frac{L^{-1/2}}{d}\right], \tag{2.1}$$

$$\left[\frac{dL}{dt}\right]_d = \kappa L^2 \phi_d \left[\frac{L^{-1/2}}{d}\right]. \tag{2.2}$$

From a microscopic point of view, the reduction of both production and destruction terms may be attributed to the pinning of vortices on small irregularities of the walls. This may have two different opposite contributions: due to the tendency of vortices to remain pinned on protuberances of the walls, the walls would reduce the rate of formation of the vortices in the flow, as compared with the same volume of the fluid in the absence of the wall; on the other side, the fact that, once pinned on the walls, the vortices become more resistent to elimination would imply a reduction in the rate of destruction.

3. Microscopic picture of the transitions and choice of the corrective terms

To choose the form of the corrective functions, in this section, we propose a microscopic scenario of the two transitions. 1) *Laminar regime ($V < V_{c1}$).* In the channel are present "remnant vortices", which are strongly pinned to protuberances of the walls; as the counterflow velocity grows, these lines are bent and increase their length. Helical waves propagate in these vortices which remain pinned on the wall. 2) *Turbulence TI ($V_{c1} < V < V_{c2}$).* When V reaches the first critical velocity V_{c1}, in correspondence of these waves, small localized arrays of quantized vortices appear, which result polarized in the (mean) direction of the equilibrium configuration of the initial vortex. 3) *Turbulence TII ($V > V_{c2}$).* When V reaches the second critical velocity V_{c2}, the flow undergoes a transition to the fully developed turbulent regime TII. The critical velocity V_{c2} indicates the definitive breakdown of these localized polarizations and the transition to the homogeneous state TII.

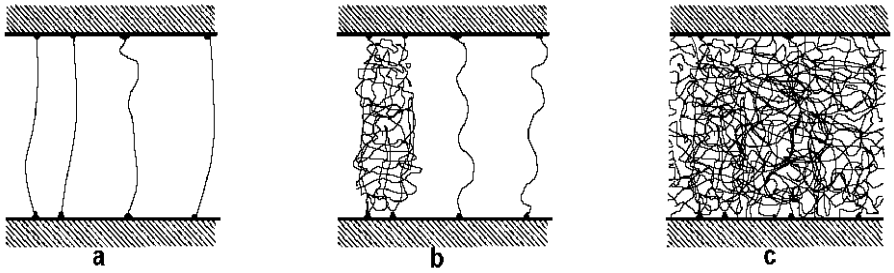

FIGURE 3.1: *A sketch of how the vortices behave, following the discussion in Sect. 2: a) laminar regime; b) turbulent TI regime ($V_{c1} < V < V_{c2}$); c) turbulent TII regime ($V > V_{c2}$). From Ref. [12].*

The transition from the laminar to the turbulent TI regime is a second order one. To describe this transition, dominated by nonlinear and nonlocal effects, we choose for ϕ_f and ϕ_d a quadratic expansion on their arguments. Precisely, we propose:

$$\frac{dL}{dt} = \alpha V L^{3/2}\left[1-\omega\frac{L^{-1/2}}{d}\right] + \alpha'\frac{V^2}{\kappa}L - \beta\kappa L^2\left[1+\omega'\frac{L^{-1/2}}{d} - \omega''\left(\frac{L^{-1/2}}{d}\right)^2\right]. \tag{3.1}$$

The TI-TII transition is a first order one. To model this transition, we observe that in the transition region there is a competition between order and disorder: the presence of the walls (terms in $1/L^{1/2}d$) contributes to the order, i.e. to the decrease of vortex lines, while the presence of the counterflow (terms in $V/\kappa L^{1/2}$) contributes to the disorder, i.e. to their increase. Therefore, to model the second transition, we assume:

$$\alpha = \alpha(V,d) = \alpha_{c2}\left(1 + c\,\tanh\left[A\left(\frac{Vd}{\kappa} - C\right)\right]\right), \tag{3.2}$$

with c, C and A dimensionless constants. In fact the ratio $\frac{Vd}{\kappa}$ is just the ratio between the two quantities corresponding to the two contribution to the order and to the disorder:

$$\frac{Vd}{\kappa} = \frac{\frac{V}{\kappa L^{1/2}}}{\frac{1}{L^{1/2}d}}. \tag{3.3}$$

With this choice, the coefficient $\alpha(V,d)$, given by equation (3.2), is approximately constant in the two turbulent regimes (α_I in turbulence TI and α_{II} in turbulence TII), while it undergoes a steep change at the second critical velocity.

4. Stationary solutions of the generalized line density evolution equation

We perform the change of variables $L^{1/2}d = y$, $Vd\kappa^{-1} = x$, obtaining

$$\frac{dy}{dt} = \frac{\beta\kappa}{2d^2}\left[-y^3 + \left(H(x)x - \omega'\right)y^2 + \left(H'x^2 - \omega H(x)x + \omega''\right)y\right], \tag{4.1}$$

with $H(x) = \frac{\alpha(x)}{\beta}$ and $H' = \frac{\alpha'}{\beta}$.

The stationary solutions of equation (4.1) are the solution $y_1 = 0$, corresponding to the laminar regime ($L_1 = 0$) and the two solutions $y_2 = y_+$ and $y_3 = y_-$ of the equation:

$$-y^2 + (H(x)x - \omega')\, y + H'x^2 - \omega H(x)x + \omega'' = 0. \tag{4.2}$$

The numerical values for the coefficients appearing in equation (4.1) have been determined through a fitting of experimental data of Martin and Tough [9]. They are reported in Table 4.1. The reader interested in the details is referred to Ref. [12].

T(K)	H_I	H_{II}	H'	ω	ω'	ω''
1.5	0.0664	0.134	0.000725	5.178	3.634	25.1
1.7	0.0830	0.172	0.000731	3.560	4.259	17.3

TABLE 4.1:*Values of H_I, H_{II}, H', ω, ω' and ω'' from this work, obtained using the experimental data of Martin and Tough [9].*

In Figure 4.1 are reported the experimental data of [9] and our theoretical predictions; we have taken $A = 0.05$ at $T = 1.5$ K and $A = 0.25$ at $T = 1.7$ K.

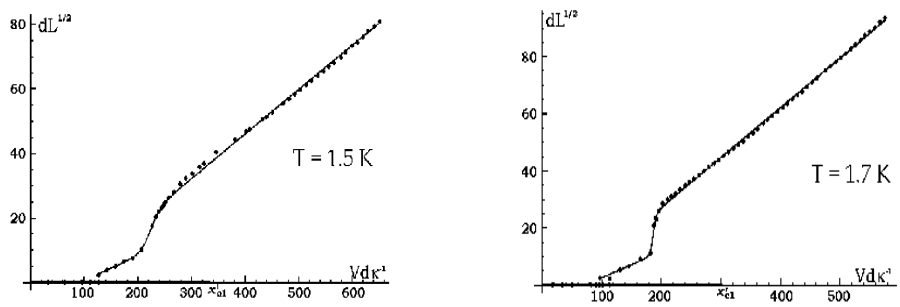

FIGURE 4.1: $y = L^{1/2}d$ *as function of* $x = Vd\kappa^{-1}$ *at* $T = 1.5$ K *and* 1.7 K. *Data are from Martin and Tough [9]. Lines are determined from this work. From Ref. [12].*

Taking in mind the simplifying hypotheses made in this Section, the good agreement between our macroscopic description and experimental observations suggests that the former one is a reasonable approximation of a theoretical unknown model, which, in the approximations made, ought to reduce itself to the equation (3.1).

5. Vortex decay toward a quiescent state

As a further illustration of the physical interest of the corrections to the decay term in (3.1), we consider the decay of vorticity in counterflow superfluid turbulence, after V is

suddenly set to zero. According to Vinen's equation (1.3), such decay is described by:

$$\frac{dL}{dt} = -\beta\kappa L^2; \qquad \text{leading to:} \qquad \frac{1}{L(t)} = \frac{1}{L_0} + \beta\kappa t. \tag{5.1}$$

This solution corresponds to the decay of an homogeneous vortex tangle. However, comparison with experimental data [23] indicates that the decay of L is much slower than this prediction. We show here how non local terms in $L^{1/2}/d$, increasingly important as L is lowered, may contribute to a slowing down of the decay. With this aim, we study decay processes using equation (3.1), which, when $V = 0$, reduces to:

$$\frac{dL}{dt} = -\beta\kappa L^2 \left[1 + \omega' \frac{L^{-1/2}}{d} - \omega'' \left(\frac{L^{-1/2}}{d}\right)^2\right]. \tag{5.2}$$

For high values of L one recovers Vinen's solution, whereas, for higher values of t, one obtains:

$$\frac{1}{L} \simeq \left(\frac{1}{L_0} - \frac{d^2}{B}\right) \exp\left(-\frac{B}{d^2}\beta\kappa t\right) + \frac{d^2}{B}, \tag{5.3}$$

where we have put $\omega''\sqrt{\omega'^2 + 4\omega''} = 2B$. The numerical values for B are reported in Table 5.1.

Finally, we observe that physical solutions of equation (5.2) have the nonvanishing asymptotic value

$$\frac{1}{dL_\infty} = \frac{\omega' + \sqrt{\omega'^2 + 4\omega''}}{2\omega''}. \tag{5.4}$$

The values of L_∞ (taking for ω' and ω'' the values reported in Table 4.1) are shown in Table 5.1. The fact that the asymptotic value of L is different from zero is satisfactory because it is experimentally known that, after the decay, a small fraction of vortices survive, pinned to the walls.

Recall now that at transition laminar $\rightarrow$ turbulence (in stationary counterflow) there is a discontinuity in the value of L, from $L = 0$ to $L_{c1}^{1/2} = y_{c1}/d$. The asymptotic values of $y = L^{1/2}d$ and the experimental values of y_{c1} determined by Martin and Tough [9] are reported in Table 5.1. As one sees there is a sufficient agreement between our values y_∞ and experimental values y_{c1}, especially at $T = 1.7$ K.

T(K)	β	y_{c1}	B	z_∞	L_∞^{-1}	$y_\infty = dL_\infty^{1/2}$
1.5	0.78	∼2.5	5.36	0.285	0.00081	3.51
1.7	1.30	∼2.5	4.70	0.393	0.00154	2.52

TABLE 5.1.*Values of β from Ref. [23] and of y_{c1} from Ref. [9]. Values of B, z_∞ and $y_\infty = L_\infty^{1/2}/d$ from this work.*

The plots of the solution of (5.2), choosing as initial point the value $1/L_0 = 1.52\ 10^{-6}\text{cm}^2$ at $T = 1.5$ K (corresponding to $y_0 = 81$) and the value $1/L_0 = 1.13\ 10^{-6}\text{cm}^2$ at $T = 1.7$ K (corresponding to $y_0 = 94$) at $T = 1.7$ K, are reported in Figure 5.1.

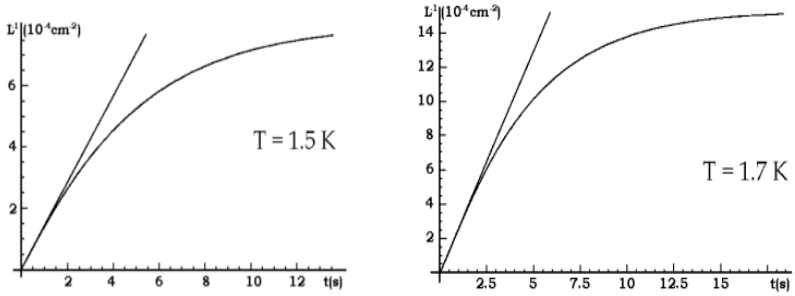

FIGURE 5.1: *Plots of the solution equation (5.2) ($1/L$ as function of t), at $T = 1.5$ K and $T = 1.7$ K, using the values of ω', ω'' and β reported in Tables 4.1 and 5.1. The initial value is $1/L_0 = 1.52\ 10^{-6}\text{cm}^2$. The straight line is solution (5.1). From Ref [12].*

Thus, the extension (3.1) of Vinen's original equation enlarges very much the ability to describe the phenomena found in superfluid counterflow experiments. In particular, it allows to describe the laminar regime ($L = 0$) including the metastability region, the transition from laminar regime to turbulent TI regime (characterized by the critical value V_{c1}^H of the velocity, and the value L_{c1} of the discontinuity in the line density) and the dependence of L with V and d for well-developed turbulence TII. The transition from TI to TII regimes is phenomenologically described introducing a steep variation in a coefficient. Further, the inclusion of corrective terms depending on δ/d in the destruction term yields a slower decay of the counterflow turbulence than Vinen's description.

6. Influence of the rotation in superfluid turbulence

Measurements of Swanson et al.[20] showed that, in a combined experiment, the effects of rotation and counterflow are not merely additive, but they exhibit some subtle nonlinear interplay, and the vortex tangle appears to be polarized by the rotation.

To derive an evolution equation for L in the presence of counterflow and rotation, motivated by the fact that the formation of vortex lines is now due to V and Ω, and taking into account the presence of channel walls, we model the formation term as:

$$\left[\frac{dL}{dt}\right]_f = \kappa L^2 \phi_f \left[\frac{V}{\kappa L^{1/2}}, \frac{\Omega^{1/2}}{(\kappa L)^{1/2}}, \frac{L^{-1/2}}{d}\right]. \tag{6.1}$$

Choosing a quadratic dependence of ϕ_f on its arguments, one obtains the following equation for the evolution of L:

$$\frac{dL}{dt} = -\beta\kappa L^2 + \left[\alpha V + \beta_2\sqrt{\kappa\Omega} - \alpha_3\frac{\kappa}{d}\right] L^{3/2}$$

$$+ \left[\alpha_2\frac{V}{d} + \beta_3\frac{\sqrt{\kappa\Omega}}{d} - \beta_1\Omega - \beta_4\frac{V\sqrt{\Omega}}{\sqrt{\kappa}} - \alpha_4\frac{\kappa}{d^2}\right] L. \tag{6.2}$$

We have chosen the four linear term depending on V and $\sqrt{\Omega}$ as production terms, while we have chosen the negative sign in the other terms which describe the nonlocal and nonlinear contributions. We have neglected here the term quadratic in V, because the values of the counterflow velocity considered are not very high.

Also in this case, we suppose that the coefficient α undergoes a steep change at the first critical velocity, putting:

$$\alpha = \alpha_{c1}\left(1 + c\,\tanh\left[N\left(\frac{\sqrt{k\Omega}}{V} - C\right)\right]\right), \tag{6.3}$$

In fact $\sqrt{\frac{\Omega}{\kappa L}}$ contributes to the order and $\frac{V}{\kappa L^{1/2}}$ to the disorder.

6.1. *Fast rotation*

In this case the terms dependent on δ/d in equation (6.2) can be neglected. One obtains therefore:

$$\frac{dL}{dt} = -\beta\kappa L^2 + \left[\alpha V + \beta_2\sqrt{\kappa\Omega}\right] L^{3/2} - \left[\beta_1\Omega + \beta_4 V\sqrt{\frac{\Omega}{\kappa}}\right] L. \tag{6.4}$$

Equation (6.4) is able to describe these experimental data. The values for the coefficients appearing in (6.4) have been obtained from the experimental values of L. They are $\alpha/\beta = 0.0473$ in the region $[0, V_{c1}^{H,R}]$ and $\alpha/\beta = 0.0469$ in the region $[V_{c1}^{H,R}, V_{c2}^{H,R}]$; $\beta_4/\beta = 0.067$, $\beta_1/\beta = 1.78$, $\alpha_2/\beta = 3.15$.

In Figure 6.1 we show the results for the vortex line density L to different values of V and Ω, with this four parameters. As one see there is a good agreement between our theoretical prediction and the experimental results. The reader interested in the details of the calculations is referred to Ref. [17].

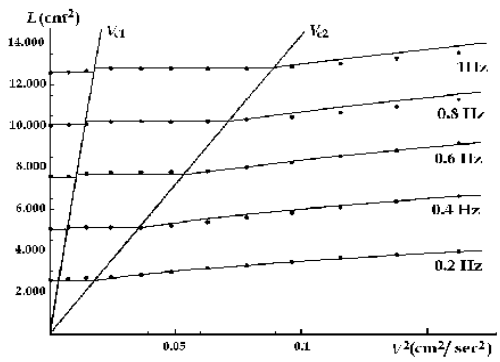

FIGURE 6.1: *Values of L as function of V^2, at various frequencies. Points are experimental data of Swanson et al. [20]*

4.2. *Slow rotation*

We have shown in [18] that, in this regime, equation (6.2) can be written as:

$$\frac{dL}{dt} = -\beta\kappa L^2 + \left[\alpha V + \beta_2\sqrt{\kappa}(\sqrt{\Omega} - \sqrt{\Omega}_c)\right] L^{3/2} - \left[\beta_4 \frac{\sqrt{\Omega} - \sqrt{\Omega}_c}{\sqrt{\kappa}} V + \beta_1(\sqrt{\Omega} - \sqrt{\Omega}_c)^2\right] L \tag{6.5}$$

where Ω_c is the critical angular velocity characterizing the appearance of the rectilinear vortex array in rotating helium II:

$$\Omega_c = \left(\frac{\beta_3}{2\beta_1}\right)^2 \frac{\kappa}{d^2}; \tag{6.6}$$

further, the following relations must be satisfied:

$$\left(\frac{\beta_3}{2\beta_1}\right)^2 = \left(\frac{\alpha_3}{\beta_2^2}\right)^2 = \left(\frac{\alpha_2}{\beta_4}\right)^2 = \frac{\alpha_4}{\beta_1}. \tag{6.7}$$

In order to establish whether the vortex-free regime is also present, we study the stability of solution $L = 0$. One obtains the following stability condition:

$$\beta_4 V + \beta_1\sqrt{\kappa}(\sqrt{\Omega} - \sqrt{\Omega}_c) \; < \; 0. \tag{6.8}$$

This inequality singles out a region of the plane $(V, \sqrt{\Omega})$ (placed in the first quadrant), delimited by a portion of straight line. Consequently V_c^H and Ω_c^R are the highest values of V and Ω respectively for which the laminar regime is present. This agrees with the experimental observation that even a very small angular velocity (but higher than the critical one), eliminates the critical counterflow velocity V_c^H.

Outside of the region of the plane $(V, \sqrt{\Omega})$ which characterizes the laminar regime, the non-zero stationary solutions of equation (6.5) are the solutions of the equation:

$$-L + \left[\frac{\alpha}{\beta\kappa} V + \frac{\beta_2}{\beta}\frac{\sqrt{\Omega} - \sqrt{\Omega}_c}{\sqrt{\kappa}}\right] L^{1/2} - \left[\frac{\beta_4}{\beta}\frac{\sqrt{\Omega} - \sqrt{\Omega}_c}{\sqrt{\kappa}} V + \frac{\beta_1}{\beta}\frac{(\sqrt{\Omega} - \sqrt{\Omega}_c)^2}{\kappa}\right] = 0. \tag{6.9}$$

A fitting with experimental data reported in Figure 1 of [27], reported also in our Figure 6.2, allows us to obtain the values for the coefficients appearing in equation (6.9), which are reported in Table 6.1.

f (Hz)	α/β	β_4/β	β_2/β	β_1/β
0.0073	0.0936	0.154	3.15	0.916
0.05	0.0843	0.139	2.22	0.816

TABLE 6.1. *Values of the coefficients appearing in equation (6.9) obtained from the data of [20].*

It is seen that, in this slow rotation regime, the coefficients α, β_4, α_2 and β_1 depend on angular velocity (or, alternatively, on anisotropy, to which we will not refer).

Though in (6.2) there are 9 parameters, corresponding to the different terms obtained from dimensional considerations, we have shown, from consistency arguments and qualitative stability trends, that only 6 of them are truly independent. For instance, we may take

β and α (the coefficients already appearing in Vinen's equation (1.3)), the three coefficients β_1, β_2, and β_4, and the coefficient β_3 to determine the values of Ω_c. The other coefficients α_2, α_3 and α_4 are determined by relations (6.7).

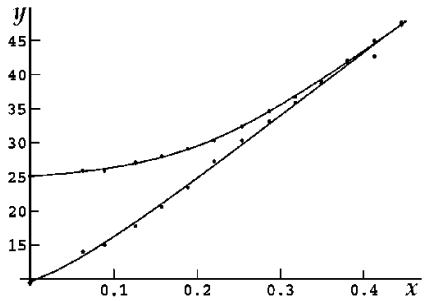

FIGURE 6.2 *Values of* $dL^{1/2}$ *as function of* Vd/κ, $a)$ *for* $f = 0.0073$ *Hz and* $b)$ *for* $f = 0.05$ *Hz from this work. Experimental data are from Ref. [20]. Plots are from Ref. [18]*

7. Conclusions and perspectives

In this paper we have reviewed several macroscopic equations for the evolution of superfluid turbulence. The central physical quantity we considered is the vortex line-density L. The starting point of our analysis has been similar to that proposed by Vinen many years ago, i.e. to provide relatively simple macroscopic equations capturing several essential features observed in the experiments and suggesting in turn new experiments to check the new predictions stemming from the equations. Once the macroscopic equation is known, one must try a qualitative and quantitative understanding of each term and of each feature suggested by the equation.

The essential equations in this paper are then the starting Vinen's equation (1.3), its extension incorporating wall effects (3.1), the formulation combining rotation and counterflow (5.4), and the extension (6.2) of such equation incorporating wall effects. All these equations, reduce to (1.3) for fully developed purely counterflow turbulence, and all of them describe the known experimental observations. It is true that they contain free parameters, but the number of them is, in any case, much less than that of the different experimental data, thus showing, indeed, the basic scientific requirements of testability.

Regarding the perspectives for the future, we would stress the need for a deeper microscopic understanding of the macroscopic terms and of the experiments. Another aspect is the insertion of the equations obtained in a thermodynamic framework. Finally, another topic would be the crossover from quantum turbulence to classical turbulence, suggesting other kinds of situations of experimental interest -as for instance grid turbulence- not considered in this paper.

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