DOI:10.1478/C1A0701004

Atti dell'Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
Vol. LXXXV, C1A0701004 (2007)
Adunanza del 15 maggio 2006

ACTIONS OF FORMAL GROUPS ON SPECIAL QUOTIENTS OF ALGEBRAS

MARILENA CRUPI * AND GAETANA RESTUCCIA

(Nota presentata dal Socio Aggregato Marilena Crupi)

ABSTRACT. Let k be a field of characteristic $p > 0$ and let F be a one dimensional commutative formal group over k. The endomorphisms of a k-algebra A that defines an action of F on A when A is isomorphic to the quotient B/pB, with B torsion free **Z**-algebra, are studied.

1. Introduction

If $F = F(X,Y)$ is a one dimensional commutative formal group over a field k and A is a k-algebra, an action D of F on A is a sequence of additive endomorphisms $\{D_i\}_{i\in N}$ of A such that $D_0 = id_A$ and $\sum_{i,j} D_i \circ D_j(a)X^iY^j = \sum_t D_t(a)F(X,Y)^t$, for every $a \in A$.

D_1 is always a derivation while the D_i's, for $i > 1$, are only additive endomorphisms such that

$$D_n(ab) = \sum_{i+j=n} D_i(a)D_j(a), \text{ for every } n.$$

If $F = F_a = X + Y$, an action of F on a k-algebra is a strongly integrable differentiation in the sense of H.Matsumura [1, 2]. In particular if $char(k) = 0$, every endomorphism D_i can be expressed in terms of D_1, for every i. In fact it is $D_i = \dfrac{1}{i!}D_1$, for every i.

If $char(k) = p > 0$ and $F = F_a$, if one considers the endomorphisms $D_1, D_p, \ldots, D_{p^i}, \ldots$, there is the nice formula [3]:

$$D_n = \frac{D_1^{n_0} D_p^{n_1} \cdots D_{p^r}^{n_r}}{n_0! n_1! \cdots n_r!}, \text{ for every } n > 0,$$

while if $F = F_m = X + Y + XY$, it is

$$D_n = \frac{(D_1)_{n_0}(D_p)_{n_1} \ldots (D_{p^r})_{n_r}}{n_0! n_1! \ldots n_r!}, \text{ for every } n > 0$$

where $(D_{p^i})_m = D_{p^i}(D_{p^i} - 1)\ldots(D_{p^i} - m + 1) = \prod_{t=0}^{m-1}(D_{p^i} - t)$ and $n = n_0 + n_1p + \cdots + n_rp^r$, $0 \leq n_i < p$, is the p-adic expansion of n [4].

The problem is the following:

Let $char(k) = p > 0$ and let F be a one dimensional commutative formal group over k. Is it possible to express every D_n in terms of $D_1, D_p, \dots, D_{p^i}, \dots$, as when $F = F_a$ and $F = F_m$?

We give a positive answer when F acts on a k-algebra A which is a quotient of a torsion free **Z**-algebra B, modulo the ideal generated by the prime number p, $p \in B$, that is $A \simeq B/pB$.

Precisely we prove the following result (Theorem **3.1**):

Let k be a separably closed field of characteristic $p > 0$, F a one dimensional commutative formal group over k of height greater than 2, and let $D : A \to A[[X]]$ be an action of F on a k-algebra A that is isomorphic to B/pB, where B is a **Z**-*algebra and $B \hookrightarrow B \otimes_{\mathbf{Z}} \mathbf{Q}$, with* **Q** *the rational number field.*

Then we have

$$D_i \circ D_{p^j} = \binom{i+p^j}{i} D_{i+p^j},$$

for $j = 0, 1$ and for every $i \in N$.

From this expression, it will be possible to express every D_n in terms of $D_1, D_p, \dots, D_{p^i}, \dots$ (Corollary **3.3**). Our result uses in a crucial way the structure theorem of one dimensional formal group over a separably closed field ([5], Chap. III, §2, Theorem 2).

2. Preliminaries

All rings are assumed to be commutative with a unit element. A local ring is assumed to be noetherian.

We recall that a one dimensional commutative formal group F over a ring k is a formal series $F(X, Y) \in k[[X, Y]]$ such that

i) $F(X, 0) = X$, $F(0, Y) = Y$,
ii) $F(F(X, Y), Z) = F(X, F(Y, Z))$
iii) $F(X, Y) = F(Y, X)$

For simplicity a one dimensional commutative formal group over a ring k will be called a formal group over k.

The most known formal groups are the additive formal group $F_a = X + Y$ and the multiplicative formal group $F_m = X + Y + XY$.

An action of the formal group F on a k-algebra A is a morphism of k-algebras $D : A \to A[[X]]$ such that if $D(a) = \sum_i D_i(a)X^i$, $a \in A$, then

$$D_0 = id_A \text{ and } \sum_{i,j} D_i \circ D_j(a) X^i Y^j = \sum_t D_t(a) F(X, Y)^t,$$

or every $a \in A$.

If F and G are formal groups over k, then a homomorphism $f : F \to G$ is a power series $f(X) \in k[[X]]$ such that $f(0) = 0$ and $f(F(X, Y)) = G(f(X), f(Y))$.

A homomorphism f is said to be an isomorphism if $f'(0)$ is an invertible element in k ($f'(X) = \partial f / \partial X$).

Moreover for any formal group F there exists a unique formal power series $i(X) \in k[[X]]$ such that $i(0) = 0$ and $F(X, i(X)) = 0 = F(i(X), X)$.

Now, for later use, let us recall the notion of height of a formal group. Let $F = F(X, Y)$ be a formal group over a ring k. As $F(X, Y) = F(Y, X)$, the induction formula: $[1]_F(X) = X$, $[m]_F(X) = F([m-1]_F(X), X)$, $m \in N$, determine a sequence of endomorphisms of the group F. If $pR = 0$, then ([5], Chap. III, §3, Theorem 2) each homomorphism $f : G \to G'$ of formal groups over k can be uniquely written in the form $f(X) = f_1(X^{p^h})$, where $f_1(X) \in k[[X]]$, $f_1'(0) \neq 0$, and $h \in N \cup \infty$ ($h = \infty$, if $f = 0$). The number h is called the height of f.

Now the height of a formal group F over a field k of characteristic $p > 0$ is defined to be the height of the endomorphism $[p]_F(X)$. We denote it by $Ht(F)$.

It is easy to see that $Ht(F) \geq 1$ for any F and that $Ht(F_a) = \infty$, $Ht(F_m) = 1$. Moreover if $Ht(F) = Ht(F')$ then $F \simeq F'$.

Let k be a separably closed field of characteristic p>0, we want to study the action of a formal group F over k on a restricted class A of k-algebras. More precisely we will study k-algebras such that $A \simeq B/pB$, where B is a $\mathbf{Z}$-algebra and $B \hookrightarrow B \otimes \mathbf{Q}$, $\mathbf{Z}$ is the ring of integers and $\mathbf{Q}$ is the field of rationals.

We recall the following:

Theorem 2.1. *([6], Lemme 19.7.1.3) Let (A, m, K) be a local $\mathbf{Z}$-algebra with $char(K) = p > 0$ and let B_0 be a K-algebra which is a regular complete local ring. Then there exists a topological local A - algebra B with respect to the topology given by the maximal ideal and such that*

i) B is a complete ring which is a flat A - module
ii) B_0 is K-isomorphic to $B \otimes_A K = B/mB$.

The previous theorem is due to Grothendieck. It is interesting to look at it because thanks to it we can have example of the algebras A considered during the paper.

Example 2.2. Let $\mathbf{Z}_p = (\mathbf{Z}, p\mathbf{Z})^\wedge$ be the local complete ring of the p-adic integers, with $p \in \mathbf{Z}$, p a prime number. $\mathbf{Z}_p$ is a topological ring with respect to the $p\mathbf{Z}_p$-adic topology.

Set $B = \mathbf{Z}_p[[X]]$, B is a complete local ring whose maximal ideal is $(p\mathbf{Z}_p, X)B$ and B is a topological ring with respect to the $(p\mathbf{Z}_p, X)B$-adic topology. Since the standard injection $\mathbf{Z}_p \hookrightarrow \mathbf{Z}_p[[X]]$ is a local continous ring homomorphism, it follows that B is a topological $\mathbf{Z}_p$-algebra which is a flat $\mathbf{Z}_p$-module.

Moreover put $B_0 = F_p[[X]]$, where $F_p = \mathbf{Z}_p/p\mathbf{Z}_p$ is the prime field, which is a perfect field, it is
$$B_0 = \mathbf{Z}_p/p\mathbf{Z}_p[[X]] \simeq \mathbf{Z}_p/p\mathbf{Z}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[X]].$$
Since $B = \mathbf{Z}_p[[X]]$ is a flat $\mathbf{Z}_p$-module, from the exactness of the sequence of $\mathbf{Z}_p$-modules:
$$0 \to p\mathbf{Z}_p \xrightarrow{j} \mathbf{Z}_p \xrightarrow{\pi} \mathbf{Z}_p/p\mathbf{Z}_p \to 0,$$
where j is the injection and π is the standard epimorphism, the exactness of the following sequence of $\mathbf{Z}_p$-modules follows:
$$0 \to p\mathbf{Z}_p \otimes \mathbf{Z}_p[[X]] \xrightarrow{j \otimes 1_{\mathbf{Z}_p[[X]]}} \mathbf{Z}_p \otimes \mathbf{Z}_p[[X]] \xrightarrow{\pi \otimes 1_{\mathbf{Z}_p[[X]]}} \mathbf{Z}_p/p\mathbf{Z}_p \otimes \mathbf{Z}_p[[X]] \to 0,$$

($\otimes = \otimes_{\mathbf{Z}_p}$). Hence:

$$\mathbf{Z}_p/p\mathbf{Z}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[X]] \simeq \frac{\mathbf{Z}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[X]]}{p\mathbf{Z}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[X]]} \simeq \mathbf{Z}_p[[X]]/p\mathbf{Z}_p[[X]]$$

and so $B_0 \simeq \mathbf{Z}_p[[X]]/p\mathbf{Z}_p[[X]] = B/pB$.

Finally $\mathbf{Z}_p$ is a $\mathbf{Z}$-flat module, $\mathbf{Z}_p[[X]]$ is a flat $\mathbf{Z}_p$-module and B is a flat $\mathbf{Z}$-module too.

3. Formal group actions on special algebras

Our main result is the following:

Theorem 3.1. *Let k be a separably closed field of characteristic $p > 0$, F a formal group over k, and $D : A \to A[[X]]$ be an action of F on a k-algebra A.*

Suppose $A \simeq B/pB$, where B is a $\mathbf{Z}$-algebra such that $B \hookrightarrow B \otimes_{\mathbf{Z}} \mathbf{Q}$. Then

i) *if $Ht(F) \geq 2$*
$$D_i \circ D_1 = (i+1)D_{i+1},$$
for every $i \in N$;

ii) *if $Ht(F) > 2$*
$$D_i \circ D_p = \binom{i+p}{i} D_{i+p},$$
for every $i \in N$.

Proof. When F is a formal group of height $h \geq 2$, F may be replaced by a formal group $\bar{F}_h$ constructed as follows.

Consider the following power series from $\mathbf{Q}[[X,Y]]$:

$$f_h(X) = X + \sum_{r=1}^{\infty} p^{-r} X^{p^{rh}} \ (f_\infty(X) = X).$$

If f_h^{-1} is the inverse homomorphism determined by f_h, we put

$$F_h(X,Y) = f_h^{-1}(f_h(X) + f_h(Y)).$$

Then $F_h = F_h(X,Y)$ is a formal group over $\mathbf{Z}$ and

$$[p]_{F_h}(X) \equiv X^{p^h} \pmod{p\mathbf{Z}[[X]]}$$

($X^{p^\infty} = 0$) ([5], Chap. I, §3.2). Hence we can define $\bar{F}_h$ as the formal group over $k \supset \mathbf{Z}/p\mathbf{Z}$ obtained by reducing all the coefficients of F_h modulo p. It follows that $Ht(\bar{F}_h) = h = Ht(F)$ and so we can assume that the formal group F is equal to $\bar{F}_h$ when $Ht(F) \geq 2$ ([5], Chap. III, §3,Theorem 2).

Now we can prove the theorem.

Consider the formal group $F_h(X,Y)$ over $\mathbf{Z}$ and an action D of this group on the $\mathbf{Z}$-algebra B, we have

$$\sum_{i,r\geq 0} D_i \circ D_r(a) X^i Y^r = \sum_{s \geq 0} D_s(a) F_h(X,Y)^s \tag{1}$$

for all $a \in A$.

i): the proof is similar to that of ([7], Lemma 4.1). But we include it for completeness.

Differentiating both sides of (1) with respect to Y and putting $Y=0$ one obtains:

$$\sum_i D_i \circ D_1(a)X^i = \sum_s sD_s(a)F_h(X,0)^{s-1}\frac{\partial F_h(X,0)}{\partial Y} =$$

$$= \sum_s sD_s(a)\frac{\partial F_h(X,0)}{\partial Y}X^{s-1}$$

From the equality $f_h(F_h(X,Y)) = f_h(X) + f_h(Y)$, by differentiating with respect to Y, we have:

$$f_h'(X)\frac{\partial F_h}{\partial Y} = 1.$$

Hence

$$\overline{f}_h'(X)\frac{\partial \overline{F}_h}{\partial Y} = 1,$$

where $\overline{f}_h'(X)$ is obtained by reducing all the coefficients of $f_h'(X)$ modulo p. Since

$$f_h'(X) = 1 + \sum_{r=1}^{\infty} p^{r(h-1)}X^{p^{rh}-1}$$

and $h \geq 2$, $\overline{f}_h'(X) = 1$. Finally $\dfrac{\partial \overline{F}_h}{\partial Y} = 1$ and we get the stated result.
ii): Differentiating both sides of (1) p-times with respect to Y, one obtains:

(2) $$\sum_{i,r} r(r-1)\dots(r-p+1)D_i \circ D_r(a)X^iY^{r-p} =$$

$$\sum_s s(s-1)\dots(s-p+1)D_s(a)F_h(X,Y)^{s-p}\left(\frac{\partial F_h(X,Y)}{\partial Y}\right)^p$$

$$+ \text{ terms with the factor } \frac{\partial^t F_h(X,Y)}{\partial Y^t}, \qquad t \geq 2.$$

From the equality $f_h(F_h(X,Y)) = f_h(X) + f_h(Y)$, by differentiating with respect to Y, we have:

(3) $$f_h'(F_h(X,Y))\frac{\partial F_h}{\partial Y} = f_h'(Y).$$

Moreover

$$f_h'(X) = 1 + \sum_{r=1}^{\infty} p^{r(h-1)}X^{p^{rh}-1}.$$

Since $h > 2$, $h-1 > 1$ and $r(h-1) > 1$, we have

$$f_h''(X) = \sum_{r=1}^{\infty} p^{r(h-1)}(p^{rh}-1)X^{p^{rh}-2}$$

and going on we obtain:

$$f_h^{(p)}(X) = \sum_{r=1}^{\infty} p^{r(h-1)}(p^{rh}-1)\dots(p^{rh}-p+1)X^{p^{rh}-p},$$

but $h > 2$ and so $f_h''(0) = \dots = f_h^{(p)}(0) = 0$.

If we calculate (3) for $Y=0$, we have

$$f_h'(X)\frac{\partial F_h(X,0)}{\partial Y}=1. \tag{4}$$

By reducing now (4) mod p, $\dfrac{\partial \bar{F}_h(X,0)}{\partial Y}=1$, $\overline{f_h'(X)}=1$.

Differentiating (3) with respect to Y we obtain

$$f_h''(F_h(X,Y))\left(\frac{\partial F_h(X,Y)}{\partial Y}\right)^2+ \tag{5}$$

$$+f_h'(F_h(X,Y))\frac{\partial^2 F_h(X,Y)}{\partial Y^2}=f_h''(Y).$$

If we calculate (5) for $Y=0$,

$$f_h''(X)\left(\frac{\partial F_h(X,0)}{\partial Y}\right)^2+f_h'(X)\frac{\partial^2 F_h(X,0)}{\partial Y^2}=0,$$

and so

$$f_h'(X)\frac{\partial^2 F_h(X,0)}{\partial Y^2}=-f_h''(X)\left(\frac{\partial F_h(X,0)}{\partial Y}\right)^2 \tag{6}$$

Finally

$$\frac{\partial^2 F_h(X,0)}{\partial Y^2}=-f_h'(X)^{-1}f_h''(X)\left(\frac{\partial F_h(X,0)}{\partial Y}\right)^2. \tag{7}$$

Claim. $f_h''(X)$ has p^2 as a factor for $h>2$.

In fact, since $h>2$, $r(h-1)>1$ and from

$$f_h''(X)=\sum_{r=1}^{\infty}p^{r(h-1)}(p^{rh}-1)X^{p^{rh}-2},$$

we get the assertion.

By reducing (6) mod p, we have $\dfrac{\partial^2 \bar{F}_h(X,0)}{\partial Y^2}=0$, since $f_h''(X)$ contains p^2 as a factor.

In general, from (7), we obtain that $\dfrac{\partial^t F_h(X,0)}{\partial Y^t}$ contains p^2 as a factor, for $t\geq 2$, and that

$$\frac{\partial^t \bar{F}_h(X,0)}{\partial Y^t}=0,\ \ \text{for}\ \ t\geq 2.$$

Consider now (1). For $Y=0$, we obtain

$$\sum_i \left(p(p-1)\dots 1\right)D_i\circ D_p(a)X^i=$$

$$=\sum_{s\geq 0}s(s-1)(s-2)\dots(s-p+1)D_s(a)X^{s-p}\left(\frac{\partial F_h(X,0)}{\partial Y}\right)^p+$$

$$+\ \text{terms with the factor}\ \ p^2.$$

$$\sum_i p! D_i \circ D_p(a) X^i = \sum_{s \geq p} p! \binom{s}{p} D_s(a) X^{s-p} \left(\frac{\partial F_h(X,0)}{\partial Y} \right)^p +$$
$$+ \text{ terms with the factor } p^2.$$

Since we can divide by p (p is not a 0-divisor in B), we have

$$\sum_i (p-1)! D_i \circ D_p(a) X^i = \sum_{s \geq p} (p-1)! \binom{s}{p} D_s(a) X^{s-p} \left(\frac{\partial F_h(X,0)}{\partial Y} \right)^p +$$
$$+ \text{ terms with the factor } p.$$

By reducing mod p, we have:

$$\sum_{i \geq 0} D_i \circ D_p(a) X^i = \sum_{s \geq p} \binom{s}{p} D_s(a) X^{s-p}.$$

Hence

$$D_i \circ D_p = \binom{i+p}{i} D_{i+p},$$

for every $i \in N$. $\square$

Remark 3.2. Observe that $\binom{i+p}{i} \neq \binom{i}{i}$ in characteristic $p > 0$.

Corollary 3.3. *Under the same hypotheses of Theorem **3.1**, let $D : A \to A[[X]]$ be an action of F on the k-algebras A. If $D(a) = \sum_{i \geq 0} D_i(a) X^i$, put*

$$\delta_0 = D_1, \delta_1 = D_p, \delta_2 = D_{p^2}, \ldots, \delta_i = D_{p^i}, \ldots$$

we have

a) *if $Ht(F) > 2$*
   i) $\delta_i \delta_j = \delta_j \delta_i$ *for $i = 0, 1$ and $j \geq 0$,*
   ii) $\delta_i^p = 0$ *for $i = 0, 1$,*
   iii) *for every $m > 0$*

$$D_m = \frac{\delta_0^{m_0} \delta_1^{m_1} \ldots \delta_r^{m_r}}{m_0! m_1! \ldots m_r!},$$

   where $m = m_0 + m_1 p + \cdots + m_r p^r$, $0 \leq m_i < p$, is the p-adic expansion of m.

b) *if $Ht(F) = 1$*
   i') $\delta_i \delta_j = \delta_j \delta_i$ *for all $i, j \geq 0$,*
   ii') $\delta_i^p = \delta_i$ *for all i,*
   iii') *for every $m > 0$*

$$D_m = \frac{(\delta_0)_{m_0} (\delta_1)_{m_1} \ldots (\delta_r)_{m_r}}{m_0! m_1! \ldots m_r!},$$

   where $(\delta_i)_m = \delta_i(\delta_{i-1}) \ldots (\delta_{i-m+1}) = \prod_{k=0}^{m-1} (\delta_i - k)$.

Proof. If we suppose that k is separably closed, the equality $Ht(F) = Ht(F')$ imply that F is isomorphic to F'. More precisely, for any $h \in N$, there exists a formal group G (that is unique up to isomorphisms) such that $Ht(G) = h$.

Then a) follows from Theorem **3.1** and b) from [8]. $\square$

Remark 3.4. Let k be a field and let H be a finite dimensional Hopf algebra over k with comultiplication $\Delta : H \to H \otimes H$ ($\otimes = \otimes_k$), antipode $S : H \to H$, and counity $\epsilon : H \to k$.

A coaction of H on a k-algebra is a morphism of algebras $D : A \to A \otimes H$ such that $(1 \otimes \epsilon)D \simeq 1$ and $(1 \otimes \Delta)$= $(D \otimes 1)D$.

From now on let k be a field of characteristic $p > 0$.

We consider the Hopf algebra which "lives" on the coalgebras C_n = $(\sum_{s=0}^{p^n-1} ke_s, \Delta, \epsilon)$, n=0,1,..., where $\Delta(e_s) = \sum_{i+j=s} e_i \otimes e_j$ and $\epsilon(e_s) = \delta_{s,0}$ (Kroneker delta). More precisely, we say that a Hopf algebra H "lives" on C_n if H, as a coalgebra, is equal to C_n.

For example $H_n(F_a) = (C_n, M : C_n \otimes C_n \to C_n, S : C_n \to C_n, \eta : k \to C_n)$, n=0,1,..., where multiplication M is given by $M(e_i \otimes e_j) = \binom{i+j}{i} e_{i+j}$ if $i + j < p^n$ and 0, otherwise, antipode S is determined by the equalities $\sum_{i+j=s} e_i S(e_j) = \delta_{s,0}$, and the structural map η is defined by $\eta(t) = te_0$.

If we fix a natural number n we can consider the Hopf algebra H which "lives" on the algebra $H_n = k[X]/(X^{p^n})$. We say also that H is a Hopf algebra structure on H_n.

A coaction of such a Hopf algebra H on a k-algebra A is a morphism of k-algebras such that if $D(a) = \sum_i D_i(a) \otimes x^i$,$a \in A$, with $x = X + (X^{p^n})$ and $D_i : A \to A$ addive endomorphisms,then

$$D_0 = 1, \qquad \text{and} \qquad D_s(ab) = \sum_{i+j=s} D_i(a)D_j(b) \text{ for all } 0 \leq s < p^n,$$

with $a, b \in A$,i.e. $\{D_i : 0 \leq i < p^n\}$ is an higher derivation of order p^n in A.

For a given formal group F over the field k and a natural n we define the Hopf algebra structure $H_n(F)$ on the algebra $H_n = k[X]/(X^{p^n})$ as follows:

a) comultiplication $\Delta : H_n \to H_n \otimes H_n \simeq k[X, Y]/\left(X^{p^n}, Y^{p^n}\right)$ defined by $\Delta(x) = F(x, y)$, where $x = X + \left(X^{p^n}, Y^{p^n}\right)$ and $y = Y + \left(X^{p^n}, Y^{p^n}\right)$, and we identify $x = X + (X^{p^n})$ with $x = X + \left(X^{p^n}, Y^{p^n}\right)$.
b) antipode $S : H_n \to H_n$ given by $S(x) = i(x)$
c) counity $\epsilon : H_n \to k$ defined by $\epsilon(x) = 0$.

We can easily verify that if F is a formal group over k and A a k-algebra a coaction of $H_n(F)$ on A is a morphism of algebras $D : A \to A \otimes H_n(F)$ such that $D_0 = 1$ and

$$\sum_{0 \leq i,j < p^n} D_i D_j(a) \otimes x^i y^j = \sum_{0 \leq s < p^n} D_s(a) \otimes F(x, y)^s$$

for all $a \in A$.

By using the same techniques of Theorem **3.1**, the following result holds:

Let k be a separably closed field of characteristic p>0, F a formal group over k, and $D : A \to A \otimes H_n(F)$ be a coaction of $H_n(F)$ on a k-algebra A.

Suppose

i) $A \simeq B/pB$, *with B **Z**-algebra such that $B \to B \otimes \mathbf{Q}$ is injective* *ii) Ht(F)* > 2

Then

$$D_i \circ D_{p^j} = \binom{i+p^j}{i} D_{i+p^j},$$

for $j = 0, 1$ *and* $0 \leq i < p^n$.

In fact put $H_n(F) = H_n$. It is easy to verify that for any action $D : A \to A[[X]]$, $D(a) = \sum_{i\geq 0} D_i(a)X^i$ of a formal group $F(X,Y)$, the application:

$$D^{(n)} : A \to A \otimes H_n, \text{ with } D^{(n)}(a) = \sum_{0\leq i<p^n} D_i(a) \otimes x^i,$$

$x = X + (X^{p^n})$, $\Delta(x) = F(x,y)$, is a coaction of the Hopf algebra H_n on A.

Hence

$$\sum_{0\leq i<p^n} D_i \circ D_p(a) \otimes x^i = \sum_{s\geq p} \binom{s}{p} D_s(a) \otimes x^{s-p}$$

and so

$$D_i \circ D_p = \binom{i+p}{i} D_{i+p},$$

for every i, $0 \leq i < p^n$.

The case $j = 0$ follows from ([7], Lemma 4.1).

References

[1] H. Matsumura, "Integrable derivations", *Nagoya Math. J.*, **87**, 227 (1982).

[2] H. Matsumura, *Commutative Algebra* (The Benjamin/ Cummings Publ. 2nd Edition, 1982).

[3] G. Restuccia, H. Matsumura, "Integrable derivations in ring of unequal characteristic", *Nagoya Math. J.*, **93**, 173 (1984).

[4] R.M. Fossum. "Invariants and Formal group Law Actions", *Contemporary Mathematics*, **43**, (1985).

[5] A. Frohlich. *Formal groups* (Lecture Notes in Math., Springer-Verlag, New York/Berlin, 1968).

[6] A.Grothendieck, *E.G.A* (Publ. I.H.E.S., Paris, 1960).

[7] G. Restuccia, A. Tyc. "Regularity of the ring of invariants under certain actions of finite abelian Hopf algebras in characteristic p>0", *J. of Algebra*, **159, No. 2**, 347 (1993).

[8] A. Tyc, "On F-integrable actions of the restricted Lie algebra of a formal group F in characteristic p>0", *Nagoya Math. J.*, **115**, 125 (1989).

Marilena Crupi, Gaetana Restuccia
Università degli Studi di Messina
Dipartimento di Matematica
Salita Sperone, 31
98166 Messina, Italy
* **E-mail**: mcrupi@dipmat.unime.it

Presented:	May 15, 2006
Published on line on	January 31, 2007