Atti dell'Accademia Peloritana dei Pericolanti
Classe I di Scienze Fis. Mat. e Nat.
Vol. LXXXI-LXXXII, C1A0401008 (2003-04)
Adunanza del 29 aprile 2004

FROM INTEGRAL MANIFOLDS AND METRICS TO POTENTIAL MAPS

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ABSTRACT. Our paper contains two main results: (1) the integral manifolds of a distribution together with two Riemann metrics produce potential maps which are in fact least squares approximations of the starting integral manifolds; (2) the least squares energy admits extremals satisfying periodic boundary conditions.

Section 1 contains historical and bibliographical notes. Section 2 analyses some elements of the geometry produced on the jet bundle of order one by a semi-Riemann Sasaki-like metric. Section 3 describes the maximal integral manifolds of a distribution as solutions of a PDEs system of order one. Section 4 studies Poisson-like second-order prolongations of first order PDE systems and formulates the Lorentz-Udrişte World-Force Law on a suitable semi-Riemann-Lagrange manifold (the base manifold of the jet bundle of order one). Section 5 exploits the idea of least squares Lagrangians, to include the integral manifolds of a distribution into a class of extremals. Section 6 gives conditions for the existence of extremals in conditions of multi-periodicity. Section 7 refers to the canonical forms of the vertical metric d-tensor produced by a density of energy on jet bundle of order one.

1. History and background

The definition of an integral manifold of an involutive distribution mimics the corresponding definition for an integral curve of a vector field. That is why, our theory which shows that a vector field together with a Riemann metric (or two Riemann metrics) generate a uni-parameter geometric dynamics [6], [11], [13] (geodesic motion in a gyroscopic field) is now redirected to prove that an involutive distribution together with two Riemann metrics produce a multi-parameter geometric dynamics. Similar ideas on geometric dynamics were already included in [7]-[13].

Our geometric dynamics is characterized by the fact that its trajectories or parametrized sheets are extremals of a quadratic Lagrangian on the jet bundle of order one. On the other hand, it is based on the idea that any PDEs system of order one admits four Poisson-like second-order prolongations, among which one is coming from a quadratic Lagrangian

(*Lagrange prolongation*), and produces a Generalized Lorentz World-Force Law [6], [8], [10]-[13].

To develop the theory we use the jet bundle of order one [5], and the semi-Riemann geometry on this space recently described by Neagu [3], one of our PhDs. We study the natural componets of the Sasaki-like metric, the components of the non-holonomic object associated to an adapted frame, the components of the Christoffel symbols, the control PDEs system describing an integral manifold of a distribution, the Poisson prolongations of first order PDEs systems, a Ganeralized Lorentz World-Force Law, least squares Lagrangians and their associated Hamiltonians, conditions ensuring the existence of extremals, canonical forms of vertical metrics tensor, Olver [4] classification of planar vertical metrics, and we formulate open problems that are near the problems of Grassi [1].

2. Semi-Riemann structure on jet bundle of order one

Of course, a constant convention is that we work in the category of C^∞ manifolds and maps, i.e., all manifolds and maps are C^∞, unless otherwise stated.

Let (T, h) and (M, g) be semi-Riemann manifolds of dimensions p and n. Hereafter we shall assume that the manifold T is oriented. Greek (Latin) letters will be used for indexing the components of geometrical objects on the manifold T (manifold M), or on manifolds produced by T and M.

Local coordinates will be written

$$\begin{aligned} t &= (t^\alpha), \quad \alpha = 1, \ldots, p \\ x &= (x^i), \quad i = 1, \ldots, n \end{aligned}$$

and the components of the metric tensors h and g and of the Christoffel symbols will be denoted respectively by

$$h_{\alpha\beta}, g_{ij}, H^\alpha_{\beta\gamma}, G^i_{jk}.$$

Indices of geometrical objects or of distinguished geometrical objects will be rised and lowered in the usual fashion using contractions with $h^{\alpha\beta}, h_{\alpha\beta}, g^{ij}, g_{ij}$.

The jet bundle of order one $J^1(T, M)$ is a differentiable manifold of dimension $p + n + pn$. Its natural coordinates are $\left(t^\alpha, x^i, x^i_\alpha = \dfrac{\partial x^i}{\partial t^\alpha}\right)$.

Physically: t^α represent the multi-time coordinates, $x^i(t^\alpha)$ represent the field components, while x^i_α are the partial derivatives of the field. In this sense, the 1-st order jet bundle is a basic manifold in the study of the mathematics of classical and quantum field theories.

Geometrically: $t = (t^\alpha)$ is a point of T, $x^i = x^i(t^\alpha)$ is a parametrized sheet, $x^i_\alpha(t) = \dfrac{\partial x^i}{\partial t^\alpha}(t)$ are partial velocities. If the partial velocities are linearly independent throughout, then a parametrized sheet is a parametrized p-surface.

A local changing of coordinates

$$(t^\alpha, x^i, x^i_\alpha) \to (t^{\alpha'}, x^{i'}, x^{i'}_{\alpha'}),$$

on $J^1(T, M)$, is given by

$$t^{\alpha'} = t^{\alpha'}(t^\alpha),\ x^{i'} = x^{i'}(x^i),\ x^{i'}_{\alpha'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial t^\alpha}{\partial t^{\alpha'}} x^i_\alpha, \tag{1}$$

where

$$\det\left(\frac{\partial t^{\alpha'}}{\partial t^\alpha}\right) \neq 0, \quad \det\left(\frac{\partial x^{i'}}{\partial x^i}\right) \neq 0.$$

The expression of the Jacobian matrix of the local diffeomorphism (1) shows that $J^1(T, M)$ is always orientable.

The natural dual bases

$$\left(\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i_\alpha}\right), \quad (dt^\alpha, dx^i, dx^i_\alpha)$$

are not suitable for the geometry of $J^1(T, M)$ since the components of geometrical objects with respect to these bases and induced bases have too complicated laws of changing under change of coordinates. For that reason we shall use the adapted dual bases

$$\left(\frac{\delta}{\delta t^\alpha} = \frac{\partial}{\partial t^\alpha} + H^\gamma_{\alpha\beta} x^i_\gamma \frac{\partial}{\partial x^i_\beta}, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^h_{ik} x^k_\alpha \frac{\partial}{\partial x^h_\alpha}, \ \frac{\partial}{\partial x^i_\alpha}\right) \tag{2}$$

$$(dt^\beta,\ dx^j,\ \delta x^j_\beta = dx^j_\beta - H^\gamma_{\beta\lambda} x^j_\gamma dt^\lambda + G^j_{hk} x^h_\beta dx^k).$$

Duality of these frames means

$$dt^\beta\left(\frac{\delta}{\delta t^\alpha}\right) = \delta^\beta_\alpha,\ dt^\beta\left(\frac{\delta}{\delta x^i}\right) = 0,\ dt^\beta\left(\frac{\partial}{\partial x^i_\alpha}\right) = 0$$

$$dx^j\left(\frac{\delta}{\delta t^\alpha}\right) = 0,\ dx^j\left(\frac{\delta}{\delta x^i}\right) = \delta^j_i,\ dx^j\left(\frac{\partial}{\partial x^i_\alpha}\right) = 0$$

$$\delta x^j_\beta\left(\frac{\delta}{\delta t^\alpha}\right) = 0,\ \delta x^j_\beta\left(\frac{\delta}{\delta x^i}\right) = 0,\ \delta x^j_\beta\left(\frac{\partial}{\partial x^i_\alpha}\right) = \delta^j_i \delta^\alpha_\beta.$$

Using these frames we define on $J^1(T, M)$ the induced semi-Riemann Sasaki-like metric

$$S = h_{\alpha\beta} dt^\alpha \otimes dt^\beta + g_{ij} dx^i \otimes dx^j + h^{\alpha\beta} g_{ij} \delta x^i_\alpha \otimes \delta x^j_\beta,$$

$$S^{-1} = h^{\alpha\beta}\frac{\delta}{\delta t^\alpha} \otimes \frac{\delta}{\delta t^\beta} + g^{ij}\frac{\delta}{\delta x^i} \otimes \frac{\delta}{\delta x^j} + h_{\alpha\beta} g^{ij} \frac{\partial}{\partial x^i_\alpha} \otimes \frac{\partial}{\partial x^j_\beta}.$$

Remark. The metric S is not homogeneous on fibres; if we need such a metric, we modify the last term by multiplying it with $c/h^{\alpha\beta} g_{ij} x^i_\alpha x^j_\beta$, where c is a constant.

Of course we can write

$$S = S_{AB} dx^A \otimes dx^B, \quad S^{-1} = S^{AB}\frac{\partial}{\partial x^A} \otimes \frac{\partial}{\partial x^B},$$

but then we need $A, B \in \{\alpha, i, \binom{\alpha}{i}\}$ like inferior indices and $A, B \in \{\alpha, i, \binom{i}{\alpha}\}$ as superior indices.

Proposition. *1) The natural non-vanishing components S_{AB} of the (Sasaki-like semi-Riemann) metric S are*

$$S_{\alpha\beta} = h_{\alpha\beta} + h^{\lambda\mu} g_{ij} H^{\gamma}_{\lambda\alpha} x^i_\gamma H^{\delta}_{\mu\beta} x^j_\delta$$
$$S_{ij} = g_{ij} + h^{\lambda\mu} g_{hk} G^h_{li} x^l_\lambda G^k_{mj} x^m_\mu$$
$$S{\binom{\alpha}{i}}{\binom{\beta}{j}} = h^{\alpha\beta} g_{ij}$$
$$S_\lambda{\binom{\alpha}{i}} = -h^{\alpha\beta} g_{ij} H^{\gamma}_{\beta\lambda} x^j_\nu$$
$$S_k{\binom{\alpha}{i}} = h^{\alpha\beta} g_{ij} G^j_{hk} x^h_\beta$$
$$S_{\lambda k} = -h^{\alpha\beta} g_{ij} H^{\gamma}_{\alpha\lambda} x^i_\gamma G^j_{hk} x^h_\beta.$$

2) The natural non-vanishing components S^{AB} of the metric S are

$$S^{\alpha\beta} = h^{\alpha\beta}$$
$$S^{ij} = g^{ij}$$
$$S{\binom{i}{\alpha}}{\binom{j}{\beta}} = h_{\alpha\beta} g^{ij} + h^{\lambda\mu} H^{\gamma}_{\lambda\alpha} x^i_\gamma H^{\nu}_{\mu\beta} x^j_\nu + g^{hm} G^i_{hk} x^k_\alpha G^j_{ml} x^l_\beta$$
$$S^\alpha{\binom{j}{\beta}} = h^{\alpha\mu} H^{\gamma}_{\mu\beta} x^j_\gamma$$
$$S^i{\binom{j}{\beta}} = -g^{ik} G^j_{kl} x^l_\beta.$$

We can compute the Christoffel symbols determined by S_{AB}, S^{AB} but their expressions are too complicated. For that reason we prefer to find the components of these Christoffel symbols with respect to the adapted bases (2),

$$D_A = M^B_A \partial_B, \quad D_A \in \left\{ \frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^i_\alpha} \right\}$$
$$\theta^A = L^A_B dx^B, \quad \theta^A \in \{dt^\alpha, dx^i, \delta x^i_\alpha\},$$

where

$$M^A_B L^B_C = \delta^A_C, \quad M^A_B L^C_A = \delta^C_B.$$

First we need the components of the non-holonomic object W which is important when we use a frame of reference such as D_A (it is not the natural frame produced by the coordinate system). These components are defined by

$$W_{BC}{}^A = (D_B M^D_C - D_C M^D_B) L^A_D$$

or

$$[D_B, D_C] = W_{BC}{}^A D_A.$$

Consequently, the non-vanishing components of W are

$$W_{\alpha\beta}{\binom{i}{\delta}} = H_{\alpha\beta\delta}{}^\mu x^i_\mu + H^\mu_{\beta\delta} \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\mu} - H^\mu_{\alpha\delta} \frac{\partial^2 x^i}{\partial t^\beta \partial t^\mu}$$
$$W_{\alpha i}{\binom{h}{\beta}} = -G^h_{ik} \frac{\partial^2 x^k}{\partial t^\alpha \partial t^\beta}$$
$$W_\alpha{\binom{\lambda}{i}}{\binom{j}{\beta}} = -H^\beta_{\alpha\lambda} \delta^i_j$$
$$W_{ij}{\binom{h}{\alpha}} = G_{jik}{}^h x^k_\alpha$$
$$W_i{\binom{\beta}{j}}{\binom{h}{\alpha}} = G^h_{ij} \delta^\beta_\alpha,$$

where $H_{\alpha\beta\delta}{}^{\mu}$ are the components of curvature tensor field produced by $H^{\mu}_{\beta\delta}$, and $G_{jik}{}^{h}$ are the components of curvature tensor field produced by G^{h}_{ij}.

The components of the Christoffel symbols determined by the metric S, with respect to the adapted frames, are given by the formulas

$$\begin{aligned}\Gamma^{A}_{BC} &= \frac{1}{2}S^{AD}(D_B S_{DC} + D_C S_{BD} - D_D S_{BC}) + \\ &+ \frac{1}{2}(W_{BC}{}^{A} + W^{A}{}_{BC} + W^{A}{}_{CB}),\end{aligned}$$

where

$$W^{A}{}_{BC} = S^{AD}S_{EC}W_{DB}{}^{E}.$$

Open problems. 1) Develop the geometry of the sphere jet bundle (hypersurface)

$$SJ^1(T,M) : h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta = 1.$$

The normal vector field on $SJ^1(T,M)$ has the covariant components

$$\left(\frac{\partial h^{\alpha\beta}}{\partial t^\gamma}g_{ij}x^i_\alpha x^j_\beta,\ h^{\alpha\beta}\frac{\partial g_{ij}}{\partial x^k}x^i_\alpha x^j_\beta,\ 2h^{\alpha\beta}g_{ij}x^j_\beta\right).$$

2) We first notice that, in $(J^1(T,M),S)$ there exists a globally defined 1-form d-tensor

$$\omega = g_{ij}x^j_\alpha dx^i \otimes dt^\alpha.$$

Its exterior differential

$$\Omega = d\omega = (-g_{ij}dx^i \wedge \delta x^j_\alpha) \otimes dt^\alpha$$

is also globally defined 2-form d-tensor, and has the components

$$(\Omega_{\alpha\ AB}) = \begin{pmatrix} 0 & -g_{ij}\delta^\beta_\alpha \\ g_{ij}\delta^\beta_\alpha & 0 \end{pmatrix}$$

in the adapted frame. Find a suitable geometry produced by ω and Ω on $J^1(T,M)$.

If the metrics h, g are positive definite, then the metric S is positive definite. In this case, the section $t^\alpha = c^\alpha$, $\alpha = 1, \dots, p$ is an $(1+p)n$-dimensional Riemann submanifold of $J^1(T,M)$ which can be identified with the Riemann manifold $({}^p\mathcal{T}(M),\ g + h^{-1} \otimes g)$, where h has constant components, and ${}^p\mathcal{T}(M) = \bigcup\limits_{x\in M}(\mathcal{T}_x M)^p$. The closed 2-forms $\Omega_\alpha = -g_{ij}dx^i \wedge \delta x^j_\alpha$, and the metric $g + h^{-1} \otimes g$ produce an almost p-Kählerian structure on ${}^p\mathcal{T}(M)$ in the sense of Grassi [1].

3) Can we have a theory of Hamilton-Poisson systems on $J^1(T,M)$? In this sense let φ, ψ be real C^∞ functions on $J^1(T,M)$. The maps

$$\{\varphi,\psi\}_\alpha = g^{ij}h_{\alpha\beta}\left(\frac{\delta\varphi}{\delta x^i}\frac{\partial\psi}{\partial x^j_\beta} - \frac{\partial\varphi}{\partial x^i_\beta}\frac{\delta\psi}{\delta x^j}\right), \quad \alpha = 1, \dots, p$$

define a Poisson structure on $J^1(T,M)$ via the 1-form Poisson bracket

$$\{\varphi,\psi\} = \{\varphi,\psi\}_\alpha dt^\alpha.$$

Also the maps $\{\varphi,\psi\}_\alpha$ define a p-Poisson structure on $({}^p\mathcal{T}(M), g + h^{-1} \otimes g)$ compatible with the almost p-Kählerian structure $\Omega_\alpha = -g_{ij}dx^i \wedge \delta x^j_\alpha$.

3. Integral manifolds of a distribution

In the preparation of the next section we describe the maximal integral manifolds of a distribution in terms of solutions of a PDEs system of order one.

A *p-dimensional distribution* on a manifold M of dimension n is a function $\mathcal{D}$ on M such that

(1) $\mathcal{D}(x)$ is a p-dimensional subspace of $\mathcal{T}_x M$ (where $0 < p \leq n$);

(2) each point x has a neighborhood V on which vector fields Y_α, $\alpha = 1, \dots, p$, are defined so that $\mathcal{D}(x) = Span\{Y_1(x), \dots, Y_p(x)\}$.

The set $\{Y_\alpha,\ \alpha = 1, \dots, p\}$ is called a *basis* for $\mathcal{D}$ at x.

A chart of M at the point x, with the coordinates x^i, $i = 1, \dots, n$ is said to be *flat* with respect to a distribution $\mathcal{D}$ on M if the vector fields $\dfrac{\partial}{\partial x^1}, \dots, \dfrac{\partial}{\partial x^p}$ form a basis for $\mathcal{D}$. A distribution on M is called *integrable* if each point of M lies in the domain of a flat chart.

A vector field X belongs to the distribution $\mathcal{D}$ if $X(x) \in \mathcal{D}(x)$, for each point x in the domain of X. The distribution $\mathcal{D}$ is called *involutive* if $[X, Y] \in \mathcal{D}$ whenever $X, Y \in \mathcal{D}$.

Frobenius Theorem. *A distribution is integrable iff it is involutive.*

A submanifold M' of M is called (*maximal*) *integral manifold* of $\mathcal{D}$ if

$$i_*(\mathcal{T}_x M') = \mathcal{D}(x),$$

where $i : M' \to M$ is the natural injection. The dimension of M' is necessarily p, and the vectors of M which are tangent to M' belong to $\mathcal{D}$. If the distribution $\mathcal{D}$ is integrable, then a parametrization of M' is obtained as solution of the PDEs system

$$\frac{\partial x^i}{\partial t^\alpha} = X^i_\alpha(t, x(t)),\ i = 1, \dots, n;\ \alpha = 1, \dots, p \tag{3}$$

$$X^i_\alpha(t, x(t)) = C^\beta_\alpha(t) Y^i_\alpha(x(t)), \quad rank\ (C^\beta_\alpha) = p,$$

where the control matrix $(C^\beta_\alpha(t))$ is fixed by the Frobenius integrability conditions

$$\frac{\partial X^i_\alpha}{\partial t^\beta} + \frac{\partial X^i_\alpha}{\partial x^j} X^j_\beta = \frac{\partial X^i_\beta}{\partial t^\alpha} + \frac{\partial X^i_\beta}{\partial x^j} X^j_\alpha.$$

Adding to the PDEs system (3) the initial condition $x(t_0) = x_0$, the solution is unique. If we impose two points $x(t_0) = x_0$, $x(t_1) = x_1$, then we can have or not solutions satisfying such boundary conditions.

Remark. Generally, we can introduce (non-maximal) integral manifolds M' of dimensions smaller than p using $i_*(\mathcal{T}_x M') \subset \mathcal{D}(x)$.

4. Poisson prolongations of first order PDEs system

Let (T, h) and (M, g) be semi-Riemann manifolds of dimensions p and n. Then $(T \times M, h + g)$ and $(J^1(T, M), h + g + h^{-1} \otimes g)$ are semi-Riemann manifolds.

A p-dimensional distribution $\mathcal{D}$ on M becomes automatically a p-dimensional distribution on $T \times M$ and hence on $J^1(T, M)$. The suitable basis $\{X^i_\alpha,\ \ i = 1, \dots, n;$

$\alpha = 1, \ldots, p\}$ of $\mathcal{D}$ and the semi-Riemann metrics h and g determine the *potential energy*

$$f : T \times M \to R, \quad f = \frac{1}{2} h^{\alpha\beta} g_{ij} X^i_\alpha X^j_\beta$$

of the distribution $\mathcal{D}$ on $T \times M$. The basis $\{X^i_\alpha\}$, the semi-Riemann metric g, and the connection ∇ induced by g determine the external distinguished tensor field

$$F_j{}^i{}_\alpha = \nabla_j X^i_\alpha - g^{ih} g_{kj} \nabla_h X^k_\alpha$$

which describes the *helicity* of the basis. Of course, the helicity vanishes iff $X_\alpha = grad\varphi_\alpha$. The helicity of the basis, and the metric h produce on $J^1(T, M)$ the *gyroscopic vector field*

$$h^{\alpha\beta} F_j{}^i{}_\alpha x^j_\beta = \xi^i$$

which describe the helicity of the distribution on $J^1(T, M)$.

Let us show that the solutions of the PDEs system (3) are potential-like maps in a suitable geometrical structure of $T \times M$ or $J^1(T, M)$. These solutions are (p-dimensional sheets) maximal integral manifolds of $\mathcal{D}$.

First we remark that X^i_α is a distinguished tensor field on $T \times M$. Second, the derivative along a solution of the PDEs system (3),

$$\frac{\delta}{\partial t^\beta} x^i_\alpha = \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} - H^\gamma_{\alpha\beta} x^i_\gamma + G^i_{jk} x^j_\alpha x^k_\beta = x^i_{\alpha\beta}, \text{ where } x^i_\alpha = \frac{\partial x^i}{\partial t^\alpha},$$

is a d-tensor on $J^1(T, M)$.

Theorem. *The differential operator* $\frac{\delta}{\partial t^\beta}$ *and some mathematical artifices tranform each solution of PDEs system (3) into four potential-like maps.*

Proof. Using the operator $\frac{\delta}{\partial t^\beta}$, from (3) we obtain the second order prolongation

$$x^i_{\alpha\beta} = (\nabla_j X^i_\alpha) x^j_\beta + D_\beta X^i_\alpha, \tag{4}$$

where

$$\nabla_j X^i_\alpha = \frac{\partial X^i_\alpha}{\partial x^j} + G^i_{jk} X^k_\alpha, \quad D_\beta X^i_\alpha = \frac{\partial X^i_\alpha}{\partial t^\beta} - H^\gamma_{\beta\alpha} X^i_\gamma.$$

Now we pass from the PDEs system (4) to the equivalent system

$$x^i_{\alpha\beta} = g^{ih} g_{kj} (\nabla_h X^k_\alpha) x^j_\beta + F_j{}^i{}_\alpha x^j_\beta + D_\beta X^i_\alpha, \tag{4'}$$

adding and subtrating a convenient term.

Succesively we modify the PDEs system (4′) replacing x^j_β with X^j_β, as follows

$$x^i_{\alpha\beta} = g^{ih} g_{kj} (\nabla_h X^k_\alpha) X^j_\beta + F_j{}^i{}_\alpha x^j_\beta + D_\beta X^i_\alpha \tag{5}$$

$$x^i_{\alpha\beta} = g^{ih} g_{kj} (\nabla_h X^k_\alpha) x^j_\beta + F_j{}^i{}_\alpha X^j_\beta + D_\beta X^i_\alpha \tag{6}$$

$$x^i_{\alpha\beta} = g^{ih} g_{kj} (\nabla_h X^k_\alpha) X^j_\beta + F_j{}^i{}_\alpha X^j_\beta + D_\beta X^i_\alpha. \tag{7}$$

Taking the trace with respect to $h^{\alpha\beta}$ and denoting $\tau(\varphi)^i = h^{\alpha\beta} x^i{}_{\alpha\beta}$, we obtain the generalized Poisson-like PDEs

$$\tau(\varphi)^i = g^{ih} h^{\alpha\beta} g_{kj} (\nabla_h X^k_\alpha) x^j_\beta + h^{\alpha\beta} F_j{}^i{}_\alpha x^j_\beta + h^{\alpha\beta} D_\beta X^i_\alpha, \tag{8}$$

$$\tau(\varphi)^i = g^{ih}h^{\alpha\beta}g_{kj}(\nabla_h X^k_\alpha)X^j_\beta + h^{\alpha\beta}F_j{}^i{}_\alpha x^j_\beta + h^{\alpha\beta}D_\beta X^i_\alpha, \tag{9}$$

$$\tau(\varphi)^i = g^{ih}h^{\alpha\beta}g_{kj}(\nabla_h X^k_\alpha)x^j_\beta + h^{\alpha\beta}F_j{}^i{}_\alpha X^j_\beta + h^{\alpha\beta}D_\beta X^i_\alpha, \tag{10}$$

$$\tau(\varphi)^i = g^{ih}h^{\alpha\beta}g_{kj}(\nabla_h X^k_\alpha)X^j_\beta + h^{\alpha\beta}F_j{}^i{}_\alpha X^j_\beta + h^{\alpha\beta}D_\beta X^i_\alpha. \tag{11}$$

Of course, the second order PDE systems (4) - (7) or (8) - (11) are prolongations of the first order PDEs system (3). They are called *Poisson-like prolongations* of (3).

A solution of generalized Poisson-like PDEs is called *potential-like map*. To avoid the misunderstandings we point out that the present terminology is different from those in [8].

Each of systems (8) - (11) is connected to the multi-time dynamics created by the distribution $\mathcal{D}$ or to the multi-time dynamics of a particle which is sensitive to the distribution $\mathcal{D}$. Since

$$g^{ih}h^{\alpha\beta}g_{kj}(\nabla_h X^k_\alpha)X^j_\beta = (grad\, f)^i,$$

the Lagrange prolongation (9) describes a generalized Lorentz World-Force Law attached to the PDEs system (3), and to the metrics h and g. From this point of view, the solutions of PDE systems (8), (10), (11) are subjects for further research.

Definition. Let $F_\alpha = (F_j{}^i{}_\alpha)$ and $U_{\alpha\beta} = (U^i_{\alpha\beta})$ be C^∞-distinguished tensors on $T \times M$, such that $\omega_{ji\alpha} = g_{hi}F_j{}^h{}_\alpha$ is skew-symmetric with respect to j and i. Let $c(t,x)$ be a C^∞ real function on $T \times M$. A C^∞ map $\varphi : T \to M$ obeys the *Generalized Lorentz World-Force Law* with respect to F_α, $U_{\alpha\beta}$, c iff

$$\tau(\varphi)^i = g^{ij}\frac{\partial c}{\partial x^j} + h^{\alpha\beta}F_j{}^i{}_\alpha x^j_\beta + h^{\alpha\beta}U^i_{\alpha\beta},$$

i.e., iff φ is a potential-like map of a suitable geometrical structure.

Remark. If the metric h is positive definite, then the PDEs in this definition are of Poisson type, and consequently their solutions are potential maps.

Theorem (Lorentz-Udrişte World-Force Law). *Every solution of the PDEs system (9) is a horizontal potential-like map of the semi-Riemann-Lagrange manifold*

$$(T \times M, h + g,\ M({}^i_\alpha)_\beta = -H^\gamma_{\alpha\beta}x^i_\gamma,\ N({}^i_\alpha)_j = G^i_{jk}x^k_\alpha - F_j{}^i{}_\alpha).$$

5. Least squares Lagrangians of first order

We shall show that the PDEs system (9) describes the extremals of a quadratic Lagrangian attached to the PDEs system (3), and to the metrics h and g. In other words, we shall prove that the PDEs system (9) is a Lagrange prolongation of the PDEs system (3). The geometrical signification of the PDEs (8), (10) or (11) was not yet studied.

The boundary value problem

$$\frac{\partial x^i}{\partial t^\alpha} = X^i_\alpha(t, x(t)),\ x(t_0) = x_0,\ x(t_1) = x_1 \tag{3'}$$

can have or not solution (even if the Frobenius integrability conditions are satified identically). We shall transform this problem into a least squares problem of variational calculus as follows.

Hereafter we shall assume that the manifold T is oriented. On the jet bundle of order one $J^1(T, M)$, endowed with the semi-Riemann Sasaki-like metric S, we build the quadratic density of energy

$$E(t, x(t), x_\alpha(t)) = \frac{1}{2} h^{\alpha\beta} g_{ij} (x^i_\alpha - X^i_\alpha(t, x(t)))(x^j_\beta - X^j_\beta(t, x(t))),$$

the quadratic Lagrangian

$$L(t, x(t), x_\alpha(t)) = E(t, x(t), x_\alpha(t))\sqrt{|h|},$$

and the energy

$$E(x; T_0) = \int_{T_0} E(t, x(t), x_\alpha(t)) dv_h,$$

where $dv_h = \sqrt{|h|} dt^1 \wedge \ldots \wedge dt^p$ denotes the volume element induced by the semi-Riemann metric h, and $T_0 \subset T$ is a relatively compact domain.

If the metrics h and g are positive definite, then $E(t, x(t), x_\alpha(t))$ is the least squares density of energy, and $L(t, x(t), x_\alpha(t))$ is the least squares Lagrangian. Then it appears the problem:

(P) find $x^* \in A = \{x : T_0 \to M | x \in C^\infty_{T_0},\ x(t_0) = x_0,\ x(t_1) = x_1\}$

such that

$$E(x^*; T_0) = \min_{x \in A} E(x; T_0).$$

Consequently a solution x^* must be a critical point of the energy functional E, i.e., an extremal of the Lagrangian L which satisfy the given boundary conditions, i.e., a potential map. If there exists an integral manifold of the PDEs system $(3')$ containing the points x_0, x_1, then the potential maps containing the points x_0, x_1 approximate the integral manifold in the sense of least squares.

Theorem [8]. *1) The PDEs system (9) is the Euler-Lagrange PDEs system produced by the Lagrangian L.*

2) The Lagrangian L can be replaced by

$$L_1 = \left(\frac{1}{2} h^{\alpha\beta} g_{ij} x^i_\alpha x^j_\beta + f \right) \sqrt{|h|}$$

iff $F_j{}^i{}_\alpha = 0$.

3) If $h_{\alpha\beta}$ are constants (T is locally flat), and X^i_α did not depend explicitly on the multi-parameter t, then the PDEs system (9) is conservative, the energy-momentum tensor field being

$$T^\alpha{}_\beta = x^i_\beta \frac{\partial L}{\partial x^i_\alpha} - L\delta^\alpha_\beta.$$

Theorem. *1) The Lagrangians L and L_1 produce the same Hamiltonian*

$$H = H_1 = \left(\frac{1}{2} h^{\alpha\beta} g_{ij} x^i_\alpha x^j_\beta - f \right) \sqrt{|h|}.$$

2) The density of energy E defines the generalized impulses

$$p^\alpha_i = \frac{\partial E}{\partial x^i_\alpha} = h^{\alpha\beta} g_{ij} (x^j_\beta - X^j_\beta), \quad x^i_\alpha = \frac{\partial x^i}{\partial t^\alpha}$$

and consequently

$$H = \left(\frac{1}{2}h_{\alpha\beta}g^{ij}p_i^\alpha p_j^\beta + h_{\alpha\beta}g^{ij}p_i^\alpha X_j^\beta\right)\sqrt{|h|}.$$

3) The density of energy E_1 defines the generalized impulses

$$p_i'^\alpha = \frac{\partial E_1}{\partial x_\alpha^i} = h^{\alpha\beta}g_{ij}x_\beta^j$$

and consequently

$$H_1 = \left(\frac{1}{2}h_{\alpha\beta}g^{ij}p_i'^\alpha p_j'^\beta - f\right)\sqrt{|h|}.$$

4) On the phase space, the Hamiltonian covariant vector fields

$$\left(\frac{\partial H}{\partial p_k^\alpha}, -\frac{\partial H}{\partial x^k}\right), \quad \left(\frac{\partial H_1}{\partial p_k'^\alpha}, -\frac{\partial H_1}{\partial x'^k}\right)$$

are related via the diffeomorphism $p_i'^\alpha = p_i^\alpha + X_i^\alpha$, $x'^i = x^i$.

Proof. 1) The connection between a Lagrangian L and the associated Hamiltonian H is

$$H = x_\alpha^i\frac{\partial L}{\partial x_\alpha^i} - L. \tag{12}$$

4) It is enough to check the formula of changing the components of a vector field when we change the coordinates:

$$\left(\frac{\partial H}{\partial p_k^\alpha}, -\frac{\partial H}{\partial x^k}\right)\begin{pmatrix} \delta_\gamma^\alpha\delta_k^l & \dfrac{\partial X_k^\alpha}{\partial x^l} \\ 0 & \delta_l^k \end{pmatrix} = \left(\frac{\partial H_1}{\partial p_l'^\gamma}, -\frac{\partial H_1}{\partial x'^l}\right),$$

where

$$\frac{\partial H}{\partial p_k^\alpha} = \left(h_{\alpha\beta}g^{kj}p_j^\beta + h_{\alpha\beta}g^{kj}X_j^\beta\right)\sqrt{|h|}$$

$$\frac{\partial H}{\partial x^k} = \left(\frac{1}{2}h_{\alpha\beta}\frac{\partial g^{ij}}{\partial x^k}p_i^\alpha p_j^\beta + h_{\alpha\beta}\frac{\partial g^{ij}}{\partial x^k}p_i^\alpha X_j^\beta + h_{\alpha\beta}g^{ij}p_i^\alpha\frac{\partial X_j^\beta}{\partial x^k}\right)\sqrt{|h|}$$

$$\frac{\partial H_1}{\partial p_k'^\alpha} = \left(h_{\alpha\beta}g^{kj}p_j'^\beta\right)\sqrt{|h|}$$

$$\frac{\partial H_1}{\partial x'^k} = \left(\frac{1}{2}h_{\alpha\beta}\frac{\partial g^{ij}}{\partial x'^k}p_i'^\alpha p_j'^\beta + \frac{\partial f}{\partial x'^k}\right)\sqrt{|h|},$$

and $\begin{pmatrix} \delta_\gamma^\alpha\delta_k^l & \dfrac{\partial X_k^\alpha}{\partial x^l} \\ 0 & \delta_l^k \end{pmatrix}$ is the Jacobian matrix.

Remark. 1) On the phase space, the Hamiltonian covariant PDEs system, produced by a given Hamiltonian H, is

$$\frac{\partial x^i}{\partial t^\alpha} = \frac{\partial H}{\partial p_i^\alpha}, \quad \frac{\partial p_i^\alpha}{\partial t^\alpha} = -\frac{\partial H}{\partial x^i} \quad \text{(summation after } \alpha).$$

2) The Legendre map $\mathcal{L}_1 : p_i^\alpha = h^{\alpha\beta}g_{ij}(x_\beta^j - X_\beta^j)$, $x'^i = x^i$ transforms $J^1(T, M)$ in the dual space $J^{1*}(T, M)$, and the Legendre map $\mathcal{L}_2 : p_i'^\alpha = h^{\alpha\beta}g_{ij}x_\beta^j$, $x'^i = x^i$

in another dual space $J^{1**}(T, M)$. These dual spaces identify via the diffeomorphism $p_i'^{\alpha} = p_i^{\alpha} + X_i^{\alpha}$, $x'^i = x^i$. Consequently they are two copies of the same space.

If L is a quadratic Lagrangian, then the associated Hamiltonian can be written

$$H = \frac{1}{2} h^{\alpha\beta} g_{ij}(x_{\alpha}^i + X_{\alpha}^i)(x_{\beta}^j - X_{\beta}^j)\sqrt{|h|}.$$

Consequently this Hamiltonian corresponds either to the problem $(3')$ or to the problem

$$\frac{\partial x^i}{\partial t^{\alpha}} = -X_{\alpha}^i(t, x(t)),\ x(t_0) = x_0,\ x(t_1) = x_1.$$

In this context the manifold T must be star-like with respect to the origin.

Proposition. *If X_{α}^i are changed into $-X_{\alpha}^i$, then the Lagrangian L modifies, but the Hamiltonian H remains invariant.*

Remark. Making abstraction of $\sqrt{|h|}$, and using formula 12 we obtain the pairs:

1) $L = \frac{1}{2} h^{\alpha\beta} g_{ij} x_{\alpha}^i x_{\beta}^j - h^{\alpha\beta} g_{ij} x_{\alpha}^i X_{\beta}^j, \quad H = \frac{1}{2} h^{\alpha\beta} g_{ij} x_{\alpha}^i x_{\beta}^j;$

2) $L = \frac{1}{2} h^{\alpha\beta} g_{ij} x_{\alpha}^i x_{\beta}^j - \frac{1}{2} h^{\alpha\beta} g_{ij} X_{\alpha}^i X_{\beta}^j, \quad H = \frac{1}{2} h^{\alpha\beta} g_{ij} x_{\alpha}^i x_{\beta}^j + \frac{1}{2} h^{\alpha\beta} g_{ij} X_{\alpha}^i X_{\beta}^j;$

3) $L = \dfrac{h^{\alpha\beta} g_{ij} x_{\alpha}^i x_{\beta}^j h^{\gamma\delta} g_{kl} X_{\gamma}^k X_{\delta}^l}{h^{\lambda\mu} g_{mn} x_{\lambda}^m X_{\mu}^n},\ H = -L;$

4) $L = h^{\alpha\beta} g_{ij} x_{\alpha}^i x_{\beta}^j h^{\gamma\delta} g_{kl} X_{\gamma}^k X_{\delta}^l - (h^{\lambda\mu} g_{mn} x_{\lambda}^m X_{\mu}^n)^2$, $H = -(h^{\alpha\beta} g_{ij} x_{\alpha}^i x_{\beta}^j)^2;$

5) $L = \dfrac{h^{\alpha\beta} g_{ij} x_{\alpha}^i x_{\beta}^j h^{\gamma\delta} g_{kl} X_{\gamma}^k X_{\delta}^l}{(h^{\lambda\mu} g_{mn} x_{\lambda}^m X_{\mu}^n)^2}$, $H = -2L.$

All these functions can be considered as rational functions of

$$u = h^{\alpha\beta} g_{ij} x_{\alpha}^i x_{\beta}^j, \quad v = h^{\alpha\beta} g_{ij} x_{\alpha}^i X_{\beta}^j, \quad w = h^{\alpha\beta} g_{ij} X_{\alpha}^i X_{\beta}^j.$$

Also, imposing $h_{\alpha\beta}, g_{ij}$ to be positive definite, we obtain inequalities satisfied by L or H.

6. Closedness condition and existence of extremals

Let us discuss the existence of extremals of the energy E in the sense of calculus of variations with periodic boundary conditions. As far as I know this is still an open problem, which can be solved realising a multi-parameter version of some results in [2].

Suppose (T, h) and (M, g) are Riemann manifolds. Let $C_{T_0}^{\infty}$ be the space of indefinite differentiable functions $x : T \to M$ whose restrictions $x|_{T_0}$ are Q_{α}-periodic in $t = (t^{\alpha})$, i.e., $x|_{T_0}\left(\exp_t\left(Q_{\alpha}\frac{\partial}{\partial t^{\alpha}}\right)\right) = x|_{T_0}(t)$, where $\left\{\frac{\partial}{\partial t^{\alpha}}\right\}$ is the canonical basis of $\mathcal{T}_t T$. Suppose $g_{ij}(x)$ are P_i-periodic in $x = (x^i)$, i.e., $g_{ij}\left(\exp_x\left(P_i\frac{\partial}{\partial x^i}\right)\right) = g_{ij}(x)$, where $\left\{\frac{\partial}{\partial x^i}\right\}$ is the canonical basis of $\mathcal{T}_x M$.

Let $1 < r < \infty$. The Sobolev space $W_{T_0}^{1,r}$ is the space of functions $x \in L^r(T_0, M)$ having a weak derivative $x_* \in L^r(T_0, M)$. Of course $C_{T_0}^{\infty} \subset W_{T_0}^{1,r}$. We denote by $H_{T_0}^1$ the

Hilbert space $W^{1,2}_{T_0}$ with the inner product

$$(u, v) = \int_{T_0} \left[(\delta_{ij} u^i v^j)(t) + \left(h^{\alpha\beta} g_{ij} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial v^j}{\partial t^\beta} \right)(t) \right] dv_h .$$

Proposition. *The energy E defined on $H^1_{T_0}$ by*

$$E(x) = \int_{T_0} E(t, x(t), x_\alpha(t)) dv_h$$

is continuously differentiable and bounded from below. Moreover, every sequence (x_k) such that

$$E'(x_k) \to 0, \; E(x_k) = bounded$$

contains a convergent subsequence.

Proof. Using the previous assumptions, we obtain

$$E(x) = ||x^i_\alpha - X^i_\alpha||^2_{L^2} \geq (||x^i_\alpha||_{L^2} - ||X^i_\alpha||_{L^2})^2 \geq 0 \tag{13}$$

and consequently E is bounded from below. On the other hand E is continuously differentiable on $H^1_{T_0}$ and

$$\begin{aligned} (E'(x), v) &= \int_{T_0} \frac{\partial L}{\partial x^j}(t, x(t), x_\alpha(t)) v^j(t) dt^1 \wedge \ldots \wedge dt^p + \\ &+ \int_{T_0} \frac{\partial L}{\partial x^j_\beta}(t, x(t), x_\alpha(t)) \frac{\partial v^j}{\partial t^\beta} dt^1 \wedge \ldots \wedge dt^p . \end{aligned} \tag{14}$$

We fix a sequence (x_k) in $H^1_{T_0}$ such that $E'(x_k) \to 0$, $E(x_k) =$ bounded. From (13) it follows that the sequence $(x_{k\alpha})_{k \in N}$ is bounded in L^2, and consequently (x_k) is bounded in $H^1_{T_0}$. Going, if necessary, to a subsequence we can assume $x_k \to x$ in $H^1_{T_0}$ and $x_k \to x$ in $C(T_0, M)$. But then

$$(E'(x_k) - E'(x), x_k - x) \to 0 \quad \text{as} \quad k \to \infty.$$

The formula (14) implies

$$\begin{aligned} &(E'(x_k) - E'(x), x_k - x) = \\ &= \int_{T_0} \left(\frac{\partial L}{\partial x^i_\alpha}(t, x_k(t), x_{k\alpha}(t)) - \frac{\partial L}{\partial x^i_\alpha}(t, x(t), x_\alpha(t)) \right) (x^i_{k\alpha} - x^i_\alpha) dt^1 \wedge \ldots \wedge dt^p + \\ &+ \int_{T_0} \left(\frac{\partial L}{\partial x^i}(t, x_k(t), x_{k\alpha}(t)) - \frac{\partial L}{\partial x^i}(t, x_k(t), x_\alpha(t) \right) (x^i_k - x^i) dt^1 \wedge \ldots \wedge dt^p \geq \\ &\geq ||x^j_{k\alpha} - x^i_\alpha||^2_{L^2} + \text{other terms tending to zero when } x_k \to x. \end{aligned}$$

Then $||x^j_{k\alpha} - x^j_\alpha||_{L^2} \to 0$ as $k \to \infty$, and hence $x_k \to x$ in $H^1_{T_0}$.

Theorem. *The problem*

$$\frac{\partial L}{\partial x^k} - \frac{\partial}{\partial t^\alpha} \frac{\partial L}{\partial x^k_\alpha} = 0, \tag{15}$$

$$x(t_0) = x(t_1), \quad x^i_\alpha(t_0) = x^i_\alpha(t_1), \quad t_0, t_1 \in \partial T_0$$

has a weak solution.

Proof. By the Proposition, $c = \inf_{H^1_{T_0}} E$ is finite. Let us verify the (Palais-Smale)$_c$-condition (see [2]). We fix the sequence (x_k) in $H^1_{T_0}$ such that $E'(x_k) \to 0$ and $E(x_k) \to c$. The hypothesis

$$E\left(\exp_x\left(P_i \frac{\partial}{\partial x^i}\right)\right) = E(x), \quad 1 \leq i \leq n$$

shows that we can select $x_k \to x$ in $H^1_{T_0}$. Hence $E'(x) = 0$, $E(x) = c$, i.e., c is a critical value of E. The function E has a minimum at the point $x \in H^1_{T_0}$, at which, necessarily, $E'(x) = 0$. Since

$$(E'(x), v) = 0$$

for all $v \in C^\infty_{T_0}$, the relation (14) implies that $x^i_\alpha = \dfrac{\partial x^i}{\partial t^\alpha}$ has a weak derivative. But then $x^i_\alpha(t_1) = x^i_\alpha(t_2)$, $x^i(t_1) = x^i(t_2)$, and x is a weak solution of (15).

Theorem. *If E satisfies the (Palais-Smale)$_c$-condition, then c is a critical value of E.*

Proof. We use the sequence (x_k) such that

$$E(x_k) \to c, \quad E'(x_k) \to 0, \quad \text{as} \quad k \to \infty.$$

By the (Palais-Smale)$_c$-condition, c is a critical value.

7. Canonical forms of vertical metric tensor

Let $E : J^1(T, M) \to R$ be a density of energy. Usually, the Hessian of E with respect to the first-order derivatives (*vertical metric d-tensor field*)

$$G\binom{\alpha}{i}\binom{\beta}{j} = \frac{1}{2}\frac{\partial^2 E}{\partial x^i_\alpha \partial x^j_\beta}$$

is required to be nonsingular. This condition is equivalent to the existence of the inverse $G\binom{k}{\gamma}\binom{l}{\delta}$ satisfying the condition

$$G\binom{\alpha}{i}\binom{\beta}{j} G\binom{i}{\alpha}\binom{l}{\gamma} = \delta^\beta_\gamma \delta^l_j.$$

Sometimes is required also the positive definiteness of $G\binom{\alpha}{i}\binom{\beta}{j}$ in the sense of biquadratic biform.

Let now define some canonical forms of the metric $G\binom{\alpha}{i}\binom{\beta}{j}$.

Definition. The vertical metric d-tensor $G\binom{\alpha}{i}\binom{\beta}{j}$ is called *factorizable* if it can be written as a Kronecker product of type

$$G\binom{\alpha}{i}\binom{\beta}{j} = h^{\alpha\beta} g_{ij} \text{ or } G = h^{-1} \otimes g.$$

The factorizability is independent of the particular coordinates used.

Lemma. *Denoting $\wedge = (\alpha, \beta)$ the columns, and $I = (i, j)$ the lines, we obtain the matrix (G^Λ_I). The d-tensor field $G\binom{\alpha}{i}\binom{\beta}{j}$ is factorizable iff rank $(G^\Lambda_I) = 1$.*

Theorem. *Suppose $G\binom{\alpha}{i}\binom{\beta}{j}$ is nonsingular and factorizable.*

1) Nonsingularity of $G\binom{\alpha}{i}\binom{\beta}{j}$ is equivalent to the nonsingularity of $h^{\alpha\beta}$ and of g_{ij}.

2) If g_{ij} does not depend on x^i_α and $n \geq 2$, then $h^{\alpha\beta}$ does not depend on x^i_α. If $h^{\alpha\beta}$ does not depend on x^i_α and $p \geq 2$, then g_{ij} does not depend on x^i_α. In these hypotheses, E is a quadratic affine form with respect to x^i_α; the coefficients of the quadratic *and linear terms of E are determined by $h^{\alpha\beta}$ and g_{ij}; the free term is determined by $h^{\alpha\beta}$ and g_{ij} iff E is a perfect square.*

3) Any two of the tensors $h^{\alpha\beta}, g_{ij}, G\binom{\alpha}{i}\binom{\beta}{j}$ determine the third.

Proof. 1) We find $\det\left(G\binom{\alpha}{i}\binom{\beta}{j}\right) = (\det(h_{\alpha\beta}))^{-n}\,(\det(g_{ij}))^{p}$.

2) First we have

$$\frac{1}{2}\frac{\partial^3 E}{\partial x^i \partial x^j_\beta \partial x^k_\gamma} = \frac{\partial h^{\alpha\beta}}{\partial x^k_\gamma} g_{ij} + h^{\alpha\beta}\frac{\partial g_{ij}}{\partial x^k_\gamma}.$$

The differentiability of E, the cyclic permutation of the indices $\binom{i}{\alpha}, \binom{j}{\beta}\binom{k}{\gamma}$ and the independence of g_{ij} on x^k_γ produce

$$\frac{\partial h^{\alpha\beta}}{\partial x^k_\gamma} g_{ij} = \frac{\partial h^{\gamma\alpha}}{\partial x^j_\beta} g_{ki} = \frac{\partial h^{\beta\gamma}}{\partial x^i_\alpha} g_{jk}.$$

The contraction with g^{ij} gives

$$n\frac{\partial h^{\alpha\beta}}{\partial x^k_\gamma} = \frac{\partial h^{\gamma\alpha}}{\partial x^k_\beta}, \quad n\frac{\partial h^{\gamma\alpha}}{\partial x^j_\beta} = \frac{\partial h^{\beta\gamma}}{\partial x^j_\alpha}, \quad n\frac{\partial h^{\beta\gamma}}{\partial x^i_\alpha} = \frac{\partial h^{\alpha\beta}}{\partial x^i_\gamma}.$$

Since $n \geq 2$ it follows $\dfrac{\partial h^{\alpha\beta}}{\partial x^k_\gamma} = 0$.

Suppose $E = \dfrac{1}{2}G^{\alpha\beta}_{ij} x^i_\alpha x^j_\beta + \omega^\alpha_i x^i_\alpha + c$, i.e., $\dfrac{\partial E}{\partial x^i_\alpha} = h^{\alpha\beta} g_{ij} x^j_\beta + \omega^\alpha_i$. The d-tensor field ω^α_i does not depend on x^i_α. On the other hand we can write $\omega^\alpha_i = h^{\alpha\beta} X^j_\beta g_{ij}$, where $X^j_\beta = h_{\alpha\beta} g^{ij} \omega^\alpha_i$. The density of energy E is reduced to a perfect square iff $c = \dfrac{1}{2}h^{\alpha\beta} g_{ij} X^i_\alpha X^j_\beta$.

3) For example, $g_{ij} = \dfrac{1}{p}h_{\alpha\beta} G\binom{\alpha}{i}\binom{\beta}{j}$.

We shall write $G^{\alpha\beta}_{i\,j}$ instead of $G\binom{\alpha}{i}\binom{\beta}{j}$, and we consider a particular *case of energy*,

$$E = G^{\alpha\beta}_{i\,j} x^i_\alpha x^j_\beta, \quad \alpha, \beta = 1, 2; \quad i, j = 1, 2,$$

saying that the metric $G^{\alpha\beta}_{ij}$ is *planar*. In this case we have the following classification due to Olver [4].

Theorem. *Any planar $G^{\alpha\beta}_{i\,j}$ is equivalent to a canonical vertical fundamental d-tensor from precisely one of the following classes:*

1) $G^{11}_{11} = \pm 1,\ G^{22}_{22} = \pm 1,\ G^{22}_{11} = \alpha,\ G^{11}_{22} = \pm\alpha,\ 2G^{12}_{12} = \beta;$

2) $G^{11}_{11} = \pm 1,\ G^{22}_{22} = \pm 1,\ G^{22}_{11} = \pm 1,\ G^{12}_{12} = \beta;$

3) $G^{11}_{11} = \pm 1,\ G^{22}_{11} = \pm 1,\ 2G^{12}_{12} = 1;$

4) $G^{11}_{11} = \pm 1,\ G^{11}_{22} = \pm 1,\ 2G^{12}_{12} = 1;$

5) $G^{11}_{11} = \pm 1,\ 2G^{12}_{12} = 1;$

6) $G^{11}_{11} = 1,\ G^{11}_{22} = -1,\ 2G^{12}_{11} = 1,\ 2G^{12}_{22} = 1;$

7) $G^{11}_{11} = 1,\ G^{22}_{22} = -1,\ 2G^{11}_{12} = 1,\ 2G^{22}_{12} = 1;$

8) $G^{11}_{11} = \pm 1,\ G^{22}_{22} = \pm 1,\ G^{12}_{12} = 1,\ 2G^{22}_{12} = 1;$

9) $G^{11}_{11} = \pm 1,\ 2G^{22}_{12} = 1$;

10) $G^{11}_{11} = \pm 1,\ 2G^{12}_{22} = 1$;

11) $2G^{12}_{11} = 1,\ 2G^{11}_{12} = 1$;

12) $G^{11}_{11} = \pm 1,\ G^{22}_{22} = \pm 1$;

13) $G^{11}_{11} = \pm 1,\ G^{11}_{22} = \pm 1$;

14) $G^{11}_{11} = \pm 1$;

15) $G^{\alpha\beta}_{i\ j} = 0$,

where the non-written components are equal to zero.

Of the 15 canonical forms listed in the preceding theorem, cases 12-15 are factorizable, as well as case 1 when $\beta = 0$, $\alpha = \pm 1$, and there are an even number $(0, 2$ or $4)$ of minus signs in the energy.

Also using $\det(G^{\alpha\beta}_{i\ j})$, with (α, i)-row index and, (β, j)-column index we can decide the cases of nonsingularity in the previous list. It follows:

- case 1 is nondegenerate for $\alpha \neq 0$ and $\beta \neq -2, 2$ and degenerate otherwise;
- cases 6-8 are nondegenerate;
- cases 2-5 and 9-15 are degenerate.

Aknowledgements. Versions of this paper were presented at Michigan State University, East Lansing, Michigan, April 19, 2001, and at the Eleventh Midwest Geometry Conference, Wichita State University, Kansas, April 27-29, 2001.

Supported by Grant # 21815/28.09.98 MEN, CNCSU-31. Also, partially supported by Grant # 33784(182)/2002, CNCSIS.

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